PHYS 1301 IDEAS OF MODERN PHYSICS

Relativity Test

The following are practice questions for the test. You will need a simple calculator in the test (phones not accepted). Appended to this document is self-study material to help you master the problems.

The following formulas will be given on the test paper:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$$

$$L = L_0\sqrt{1-\left(v^2/c^2\right)}$$

$$v_{AB} = \frac{v_{AE} + v_{EB}}{1 + \frac{v_{AE}v_{EB}}{c^2}}$$

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

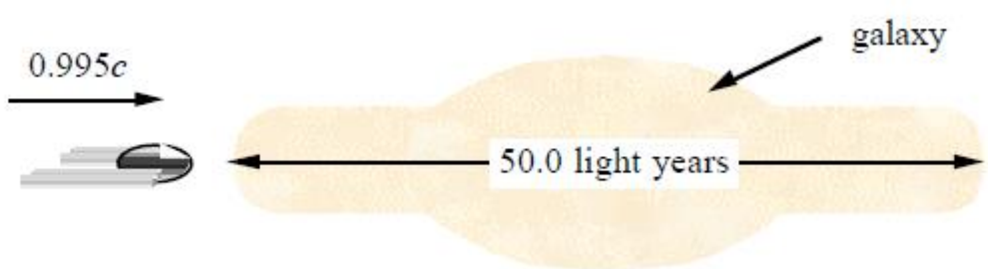
$c = 3 \times 10^8$ m/s

(m is rest mass)

1. Time Dilation

1.1 Mars rotates about its axis once every 88 642 s. A spacecraft comes into the solar system and heads directly toward Mars at a speed of 0.800c. What is the rotational period of Mars according to the beings on the spaceship?
[148 000 s]

1.2 A bomb is designed to explode 2.00 s after it is armed. The bomb is launched from earth and accelerated to an unknown final speed. After reaching its final speed, however, the bomb is observed by people on earth to explode 4.25 s after it is armed. What is the final speed of the bomb just before it explodes?
[0.882c]

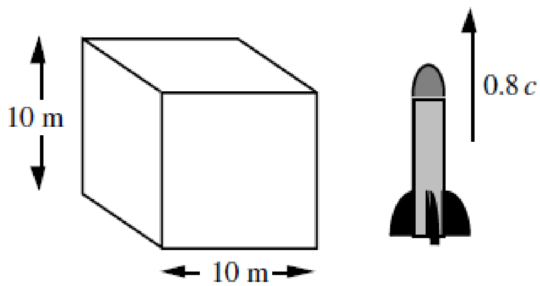


1.3 The figure shows a side view of a galaxy that is 50.0 light years in diameter (it takes light 50 years to cross it according to someone at rest in the galaxy). A spaceship enters the galactic plane with speed 0.995c relative to the galaxy. Assume that the galaxy can be treated as an inertial reference frame. How long does it take the spaceship to cross the galaxy according to a clock on board the spaceship?
[5.02 years]

2. Length Contraction

2.1 A meter stick is observed to be only 0.900 meters long to an inertial observer. At what speed, relative to the observer, must the meter stick be moving?
[1.31 x 10^8 m/s]

2.2 In question 1.3, determine the diameter of the galaxy as perceived by a person in the spaceship.
[4.99 light years]



2.3 A cubic asteroid with side length 10.0 m is in an inertial reference frame. A rocket ship moves along one side of the asteroid as shown in the figure with speed 0.80c relative to the asteroid. An astronaut in the rocket ship measures the volume of the asteroid. What volume does the astronaut measure?
[600 m^3]

3. Energy

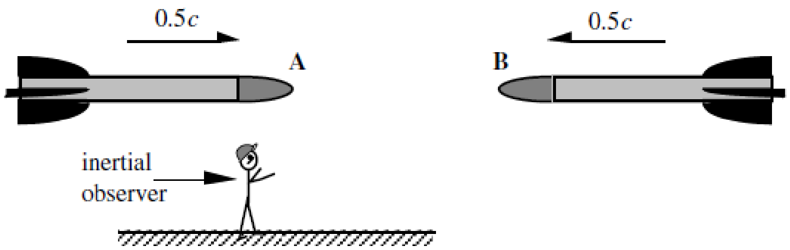
3.1 A muon particle has rest energy 105 MeV (energy unit 1 MeV = 1.6 x 10^{-13} J). What is its kinetic energy when its speed is 0.95*c*?
[231 MeV]

3.2 The average power output of a nuclear power plant is 500 Mega Watts, meaning it produces energy at a rate of 500 x 10^6 J/s. In one minute, what is the change in the mass of the nuclear fuel due to the energy being taken from the nuclear reactor? (*Assume 100% efficiency).*
[3.3 x 10^{-7} kg]

3.3 During each hour of flight, a large jet airplane consumes 3330 gallons of fuel via combustion. Combustion releases 1.2 x 10^6 Joules/gallon. One gallon of fuel has a mass of 2.84 kg. Calculate the energy equivalent of 3330 gallons of fuel and determine the ratio (in percent) of this energy equivalent to the amount of energy released by combustion in one hour of flight.
[4.69 x 10^{-10} %]

4. Relative Velocity

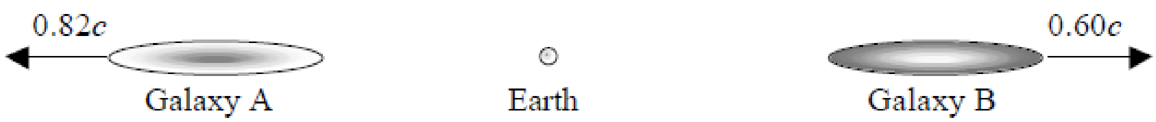
4.1 Two rockets, **A** and **B**, travel toward each other with speeds $0.5c$ relative to an inertial observer.



Determine the speed of rocket **A** relative to rocket **B**.
[$0.8c$]

4.2 A starship approaches Earth at a speed of $0.8c$ relative to the planet. On the way, it overtakes a freight ship. The relative speed of the two ships as measured by the navigator on the starship is $0.5c$. At what speed is the freight ship approaching the planet?
[$0.5c$]

4.3 Astronomers on Earth, an inertial reference frame, observe galaxies A and B that are moving away from the Earth as shown. The speeds indicated are those measured by the astronomers on Earth. What is the speed of galaxy B as measured by an observer in galaxy A?



[$0.95c$]

[redacted] We can see how this curious effect arises with the help of the clock illustrated in Figure 28.4, which uses a pulse of light to mark time. A short pulse of light is emitted by a light source, reflects from a mirror, and then strikes a detector that is situated next to the source. Each time a pulse reaches the detector, a "tick" registers on the chart recorder, another short pulse of light is emitted, and the cycle repeats. Thus, the time interval between successive "ticks" is marked by a beginning event (the firing of the light source) and an ending event (the pulse striking the detector). The source and detector are so close to each other that the two events can be considered to occur at the same location.

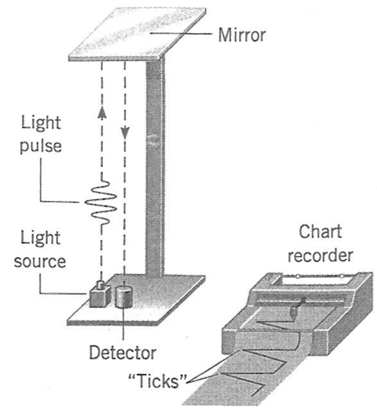


Figure 28.4 A light clock.

Suppose two identical clocks are built. One is kept on earth, and the other is placed aboard a spacecraft that travels at a constant velocity relative to the earth. The astronaut is at rest with respect to the clock on the spacecraft and, therefore, sees the light pulse move along the up/down path shown in Figure 28.5*a*. According to the astronaut, the time interval Δt_0 required for the light to follow this path is the distance $2D$ divided by the speed of light c; $\Delta t_0 = 2D/c$. To the astronaut, Δt_0 is the time interval between the "ticks" of the spacecraft clock—that is, the time interval between the beginning and ending events of the clock. An earth-based observer, however, does *not* measure Δt_0 as the time interval between these two events. Since the spacecraft is moving, the earth-based observer sees the light pulse follow the diagonal path shown in red in part *b* of the drawing. This path is longer than the up/down path seen by the astronaut. But light travels at the *same speed* c for both observers, in accord with the speed of light postulate. Therefore, the earth-based observer measures a time interval Δt between the two events that is *greater* than the time interval Δt_0 measured by the astronaut. In other words, the earth-based observer, using his own earth-based clock to measure the performance of the astronaut's clock, finds that the astronaut's clock runs slowly. This result of special relativity is known as ***time dilation.*** (To *dilate* means to expand, and the time interval Δt is "expanded" relative to Δt_0.)

The time interval Δt that the earth-based observer measures in Figure 28.5*b* can be determined as follows. While the light pulse travels from the source to the detector, the spacecraft moves a distance $2L = v\Delta t$ to the right, where v is the speed of the spacecraft relative to the earth. From the drawing it can be seen that the light pulse travels a total diagonal distance of $2s$ during the time interval Δt. Applying the Pythagorean theorem, we find that

$$2s = 2\sqrt{D^2 + L^2} = 2\sqrt{D^2 + \left(\frac{v\,\Delta t}{2}\right)^2}$$

But the distance $2s$ is also equal to the speed of light times the time interval Δt, so $2s = c\,\Delta t$. Therefore,

$$c\,\Delta t = 2\sqrt{D^2 + \left(\frac{v\,\Delta t}{2}\right)^2}$$

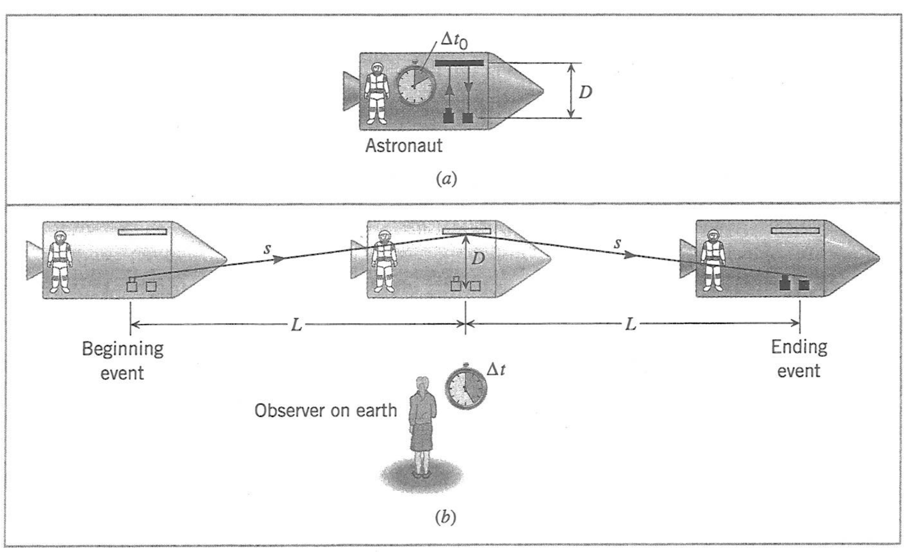


Figure 28.5 (*a*) The astronaut measures the time interval Δt_0 between successive "ticks" of his light clock. (*b*) An observer on earth watches the astronaut's clock and sees the light pulse travel a greater distance between "ticks" than it does in part *a*. Consequently, the earth-based observer measures a time interval Δt between "ticks" that is greater than Δt_0.

Squaring this result and solving for Δt gives

$$\Delta t = \frac{2D}{c} \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

But $2D/c = \Delta t_0$, the time interval between successive "ticks" of the spacecraft's clock as measured by the astronaut. With this substitution, the equation for Δt can be expressed as

Time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} \qquad (28.1)$$

The symbols in this formula are defined as follows:

Δt_0 = proper time interval, which is the interval between two events as measured by an observer who is at rest with respect to the events and who views them as occurring *at the same place*

Δt = dilated time interval, which is the interval measured by an observer who is in motion with respect to the events and who views them as occurring at *different places*

v = relative speed between the two observers

c = speed of light in a vacuum

For a speed v that is less than c, the term $\sqrt{1 - v^2/c^2}$ in Equation 28.1 is less than 1, and the dilated time interval Δt is greater than Δt_0. Example 1 shows this time dilation effect.

Example 1 Time Dilation

The spacecraft in Figure 28.5 is moving past the earth at a constant speed v that is 0.92 times the speed of light. Thus, $v = (0.92)(3.0 \times 10^8 \text{ m/s})$, which is often written as $v = 0.92c$. The astronaut measures the time interval between successive "ticks" of the spacecraft clock to be $\Delta t_0 = 1.0$ s. What is the time interval Δt that an earth observer measures between "ticks" of the astronaut's clock?

Reasoning Since the clock on the spacecraft is moving relative to the earth observer, the earth observer measures a greater time interval Δt between "ticks" than does the astronaut, who is at rest relative to the clock. The dilated time interval Δt can be determined from the time dilation relation, Equation 28.1.

Solution The dilated time interval is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{1.0 \text{ s}}{\sqrt{1 - \left(\dfrac{0.92c}{c}\right)^2}} = \boxed{2.6 \text{ s}}$$

From the point of view of the earth-based observer, the astronaut is using a clock that is running slowly, because the earth-based observer measures a time between "ticks" that is longer (2.6 s) than what the astronaut measures (1.0 s).

***The physics of* the Global Positioning System and special relativity.**

Present-day spacecrafts fly nowhere near as fast as the craft in Example 1. Yet circumstances exist in which time dilation can create appreciable errors if not accounted for. The Global Positioning System (GPS), for instance, uses highly accurate and stable atomic clocks on board each of 24 satellites orbiting the earth at speeds of about 4000 m/s. These clocks make it possible to measure the time it takes for electromagnetic waves to travel from a satellite to a ground-based GPS receiver. From the speed of light and the times measured for signals from three or more of the satellites, it is possible to locate the position of the receiver (see Section 5.5). The stability of the clocks must be better than one part in 10^{13} to ensure the positional accuracy demanded of the GPS. Using Equation 28.1 and the speed of the GPS satellites, we can calculate the difference between the dilated time interval and the proper time interval as a fraction of the proper time interval and compare the result to the stability of the GPS clocks:

$$\frac{\Delta t - \Delta t_0}{\Delta t_0} = \frac{1}{\sqrt{1 - v^2/c^2}} - 1 = \frac{1}{\sqrt{1 - (4000 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}} - 1$$

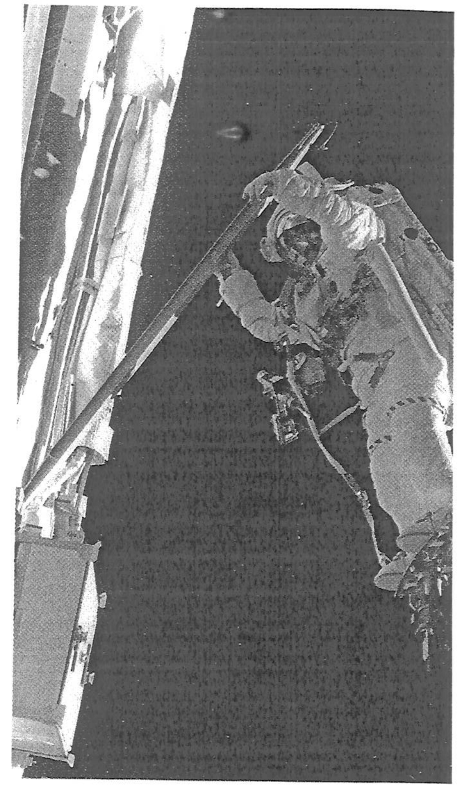
$$= \frac{1}{1.1 \times 10^{10}}$$

This result is approximately one thousand times greater than the GPS-clock stability of one part in 10^{13}. Thus, if not taken into account, time dilation would cause an error in the measured position of the earth-based GPS receiver roughly equivalent to that caused by a thousand-fold degradation in the stability of the atomic clocks.

PROPER TIME INTERVAL

In Figure 28.5 both the astronaut and the person standing on the earth are measuring the time interval between a beginning event (the firing of the light source) and an ending event (the light pulse striking the detector). For the astronaut, who is at rest with respect to the light clock, the two events occur at the same location. (Remember, we are assuming that the light source and detector are so close together that they are considered to be at the same place.) Being at rest with respect to a clock is the usual or "proper" situation, so the time interval Δt_0 measured by the astronaut is called the ***proper time interval.*** In general, the proper time interval Δt_0 between two events is the time interval measured by an observer who is at rest relative to the events and sees them at the *same location* in space. On the other hand, the earth-based observer does not see the two events occurring at the same location in space, since the spacecraft is in motion. The time interval Δt that the earth-based observer measures is, therefore, not a proper time interval in the sense that we have defined it.

To understand situations involving time dilation, it is essential to distinguish between Δt_0 and Δt. It is helpful if one first identifies the two events that define the time interval. These may be something other than the firing of a light source and the light pulse striking a detector. Then determine the reference frame in which the two events occur at the same place. An observer at rest in this reference frame measures the proper time interval Δt_0.



Shown here in orbit is astronaut Lee M. E. Morin on April 13, 2002, as he works on the International Space Station. His feet are secured to the end of the station's robotic arm. (© AP/Wide World Photos)

SPACE TRAVEL

One of the intriguing aspects of time dilation occurs in conjunction with space travel. Since enormous distances are involved, travel to even the closest star outside our solar system would take a long time. However, as the following example shows, the travel time can be considerably less for the passengers than one might guess.

Example 2 Space Travel

The physics of **space travel and special relativity.**

Alpha Centauri, a nearby star in our galaxy, is 4.3 light-years away. This means that, as measured by a person on earth, it would take light 4.3 years to reach this star. If a rocket leaves for Alpha Centauri and travels at a speed of $v = 0.95c$ relative to the earth, by how much will the passengers have aged, according to their own clock, when they reach their destination? Assume that the earth and Alpha Centauri are stationary with respect to one another.

Reasoning The two events in this problem are the departure from earth and the arrival at Alpha Centauri. At departure, earth is just outside the spaceship. Upon arrival at the destination, Alpha Centauri is just outside. Therefore, relative to the passengers, the two events occur at the same place—namely, just outside the spaceship. Thus, the passengers measure the proper time interval Δt_0 on their clock, and it is this interval that we must find. For a person left behind on earth, the events occur at *different places,* so such a person measures the dilated time interval Δt rather than the proper time interval. To find Δt we note that the time to travel a given distance is inversely proportional to the speed. Since it takes 4.3 years to traverse the distance between earth and Alpha Centauri at the speed of light, it would take even longer at the slower speed of $v = 0.95c$. Thus, a person on earth measures the dilated time interval to be $\Delta t = (4.3 \text{ years})/0.95 = 4.5$ years. This value can be used with the time-dilation equation to find the proper time interval Δt_0.

Problem solving insight
In dealing with time dilation, decide which interval is the proper time interval as follows: (1) Identify the two events that define the interval. (2) Determine the reference frame in which the events occur at the same place; an observer at rest in this frame measures the proper time interval Δt_0.

Solution Using the time-dilation equation, we find that the proper time interval by which the passengers judge their own aging is

$$\Delta t_0 = \Delta t\sqrt{1 - \frac{v^2}{c^2}} = (4.5 \text{ years})\sqrt{1 - \left(\frac{0.95c}{c}\right)^2} = \boxed{1.4 \text{ years}}$$

Thus, the people aboard the rocket will have aged by only 1.4 years when they reach Alpha Centauri, and not the 4.5 years an earthbound observer has calculated.

VERIFICATION OF TIME DILATION

A striking confirmation of time dilation was achieved in 1971 by an experiment carried out by J. C. Hafele and R. E. Keating.* They transported very precise cesium-beam atomic clocks around the world on commercial jets. Since the speed of a jet plane is considerably less than c, the time-dilation effect is extremely small. However, the atomic clocks were accurate to about $\pm 10^{-9}$ s, so the effect could be measured. The clocks were in the air for 45 hours, and their times were compared to reference atomic clocks kept on earth. The experimental results revealed that, within experimental error, the readings on the clocks on the planes were different from those on earth by an amount that agreed with the prediction of relativity.

The behavior of subatomic particles called *muons* provides additional confirmation of time dilation. These particles are created high in the atmosphere, at altitudes of about 10 000 m. When at rest, muons exist only for about 2.2×10^{-6} s before disintegrating. With such a short lifetime, these particles could never make it down to the earth's surface, even traveling at nearly the speed of light. However, *a large number of muons do reach the earth.* The only way they can do so is to live longer because of time dilation, as Example 3 illustrates.

Example 3 The Lifetime of a Muon

The average lifetime of a muon at rest is 2.2×10^{-6} s. A muon created in the upper atmosphere, thousands of meters above sea level, travels toward the earth at a speed of $v = 0.998c$. Find, on the average, (a) how long a muon lives according to an observer on earth, and (b) how far the muon travels before disintegrating.

Reasoning The two events of interest are the generation and subsequent disintegration of the muon. When the muon is at rest, these events occur at the same place, so the muon's average (at rest) lifetime of 2.2×10^{-6} s is a proper time interval Δt_0. When the muon moves at a speed $v = 0.998c$ relative to the earth, an observer on the earth measures a dilated lifetime Δt that is given by Equation 28.1. The average distance x traveled by a muon, as measured by an earth observer, is equal to the muon's speed times the dilated time interval.

Solution

(a) The observer on earth measures a dilated lifetime given by

Problem solving insight
The proper time interval Δt_0 is always shorter than the dilated time interval Δt.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6}\ \text{s}}{\sqrt{1 - \left(\frac{0.998c}{c}\right)^2}} = \boxed{35 \times 10^{-6}\ \text{s}} \qquad (28.1)$$

(b) The distance traveled by the muon before it disintegrates is

$$x = v\,\Delta t = (0.998)(3.00 \times 10^8\ \text{m/s})(35 \times 10^{-6}\ \text{s}) = \boxed{1.0 \times 10^4\ \text{m}}$$

Thus, the dilated, or extended, lifetime provides sufficient time for the muon to reach the surface of the earth. If its lifetime were only 2.2×10^{-6} s, a muon would travel only 660 m before disintegrating and could never reach the earth.

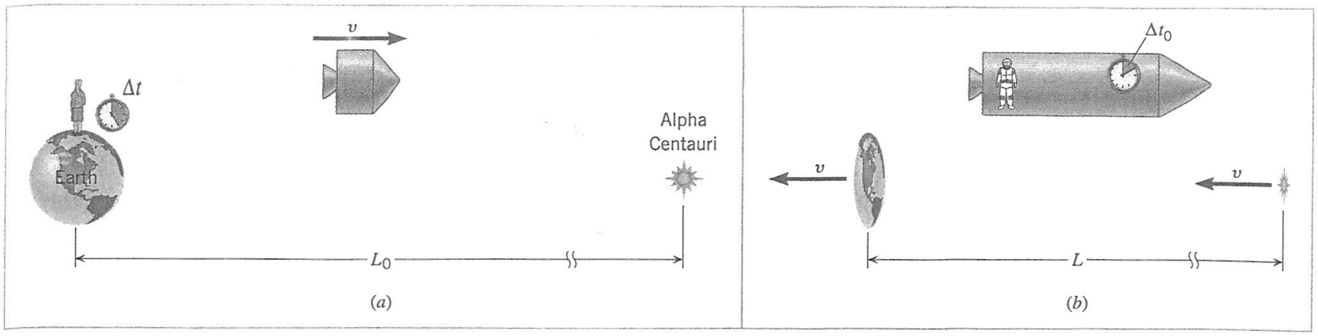


Figure 28.6 (*a*) As measured by an observer on the earth, the distance to Alpha Centauri is L_0, and the time required to make the trip is Δt. (*b*) According to the passenger on the spacecraft, the earth and Alpha Centauri move with speed v relative to the craft. The passenger measures the distance and time of the trip to be L and Δt_0, respectively, both quantities being less than those in part *a*.

28.4 The Relativity of Length: Length Contraction

Because of time dilation, observers moving at a constant velocity relative to each other measure different time intervals between two events. For instance, Example 2 in the previous section illustrates that a trip from earth to Alpha Centauri at a speed of $v = 0.95c$ takes 4.5 years according to a clock on earth, but only 1.4 years according to a clock in the rocket. These two times differ by the factor $\sqrt{1 - v^2/c^2}$. Since the times for the trip are different, one might ask whether the observers measure different distances between earth and Alpha Centauri. The answer, according to special relativity, is yes. After all, both the earth-based observer and the rocket passenger agree that the relative speed between the rocket and earth is $v = 0.95c$. Since speed is distance divided by time and the time is different for the two observers, it follows that the distances must also be different, if the relative speed is to be the same for both individuals. Thus, the earth observer determines the distance to Alpha Centauri to be $L_0 = v\Delta t = (0.95c)(4.5 \text{ years}) = 4.3$ light-years. On the other hand, a passenger aboard the rocket finds the distance is only $L = v\Delta t_0 = (0.95c)(1.4 \text{ years}) = 1.3$ light-years. The passenger, measuring the shorter time, also measures the shorter distance. This shortening of the distance between two points is one example of a phenomenon known as ***length contraction.***

The relation between the distances measured by two observers in relative motion at a constant velocity can be obtained with the aid of Figure 28.6. Part *a* of the drawing shows the situation from the point of view of the earth-based observer. This person measures the time of the trip to be Δt, the distance to be L_0, and the relative speed of the rocket to be $v = L_0/\Delta t$. Part *b* of the drawing presents the point of view of the passenger, for whom the rocket is at rest, and the earth and Alpha Centauri appear to move by at a speed v. The passenger determines the distance of the trip to be L, the time to be Δt_0, and the relative speed to be $v = L/\Delta t_0$. Since the relative speed computed by the passenger equals that computed by the earth-based observer, it follows that $v = L/\Delta t_0 = L_0/\Delta t$. Using this result and the time-dilation equation, Equation 28.1, we obtain the following relation between L and L_0:

Length contraction

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{28.2}$$

The length L_0 is called the ***proper length;*** it is the length (or distance) between two points *as measured by an observer at rest with respect to them.* Since v is less than c, the term $\sqrt{1 - v^2/c^2}$ is less than 1, and L is less than L_0. It is important to note that this length contraction occurs only along the direction of the motion. Those dimensions that are perpendicular to the motion are not shortened, as the next example discusses.

Example 4 The Contraction of a Spacecraft

An astronaut, using a meter stick that is at rest relative to a cylindrical spacecraft, measures the length and diameter of the spacecraft to be 82 and 21 m, respectively. The spacecraft moves with a constant speed of $v = 0.95c$ relative to the earth, as in Figure 28.6. What are the dimensions of the spacecraft, as measured by an observer on earth?

Reasoning The length of 82 m is a proper length L_0, since it is measured using a meter stick that is at rest relative to the spacecraft. The length L measured by the observer on earth can be determined from the length-contraction formula, Equation 28.2. On the other hand, the diameter of the spacecraft is perpendicular to the motion, so the earth observer does not measure any change in the diameter.

Solution The length L of the spacecraft, as measured by the observer on earth, is

Problem solving insight
The proper length L_0 is always larger than the contracted length L.

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = (82\text{ m})\sqrt{1 - \left(\frac{0.95c}{c}\right)^2} = \boxed{26\text{ m}}$$

Both the astronaut and the observer on earth measure the same value for the diameter of the spacecraft: $\boxed{\text{Diameter} = 21\text{ m}}$. Figure 28.6*a* shows the size of the spacecraft as measured by the earth observer, and part *b* shows the size measured by the astronaut.

When dealing with relativistic effects we need to distinguish carefully between the criteria for the proper time interval and the proper length. The proper time interval Δt_0 between two events is the time interval measured by an observer who is at rest relative to the events and sees them occurring at the *same place.* All other moving inertial observers will measure a larger value for this time interval. The proper length L_0 of an object is the length measured by an observer who is *at rest* with respect to the object. All other moving inertial observers will measure a shorter value for this length. The observer who measures the proper time interval may not be the same one who measures the proper length. For instance, Figure 28.6 shows that the astronaut measures the proper time interval Δt_0 for the trip between earth and Alpha Centauri, whereas the earth-based observer measures the proper length (or distance) L_0 for the trip.

It should be emphasized that the word "proper" in the phrases proper time and proper length does *not* mean that these quantities are the correct or preferred quantities in any absolute sense. If this were so, the observer measuring these quantities would be using a preferred reference frame for making the measurement, a situation that is prohibited by the relativity postulate. According to this postulate, there is no preferred inertial reference frame. When two observers are moving relative to each other at a constant velocity, each measures the other person's clock to run more slowly than his own, and each measures the other person's length, along that person's motion, to be contracted.

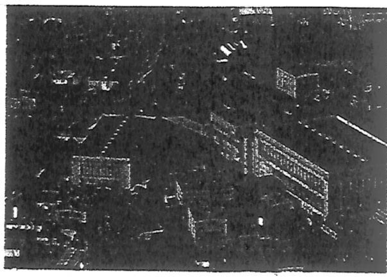


Figure 28.8 The Stanford three-kilometer linear accelerator accelerates electrons almost to the speed of light. (© Bill Marsh/Photo Researchers)

28.6 The Equivalence of Mass and Energy

THE TOTAL ENERGY OF AN OBJECT

One of the most astonishing results of special relativity is that mass and energy are equivalent, in the sense that a gain or loss of mass can be regarded equally well as a gain or loss of energy. Consider, for example, an object of mass m traveling at a speed v. Einstein showed that the ***total energy*** E of the moving object is related to its mass and speed by the following relation:

Total energy of an object

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{28.4}$$

To gain some understanding of Equation 28.4, consider the special case in which the object is at rest. When $v = 0$ m/s, the total energy is called the ***rest energy*** E_0, and Equation 28.4 reduces to Einstein's now-famous equation:

Rest energy of an object

$$E_0 = mc^2 \tag{28.5}$$

The rest energy represents the energy equivalent of the mass of an object at rest. As Example 6 shows, even a small mass is equivalent to an enormous amount of energy.

Example 6 The Energy Equivalent of a Golf Ball

A 0.046-kg golf ball is lying on the green. (a) Find the rest energy of the golf ball. (b) If this rest energy were used to operate a 75-W light bulb, for how many years could the bulb stay on?

Reasoning The rest energy E_0 that is equivalent to the mass m of the golf ball is found from the relation $E_0 = mc^2$. The power used by the bulb is 75 W, which means that it consumes 75 J of energy per second. If the entire rest energy of the ball were available for use, the bulb could stay on for a time equal to the rest energy divided by the power.

Solution

(a) The rest energy of the ball is

$$E_0 = mc^2 = (0.046 \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 = \boxed{4.1 \times 10^{15} \text{ J}} \tag{28.5}$$

(b) This rest energy can keep the bulb burning for a time t given by

$$t = \frac{\text{Rest energy}}{\text{Power}} = \frac{4.1 \times 10^{15} \text{ J}}{75 \text{ W}} = 5.5 \times 10^{13} \text{ s} \tag{6.10b}$$

Since one year contains 3.2×10^7 s, we find $\boxed{t = 1.7 \times 10^6 \text{ yr}}$, or 1.7 million years!

When an object is accelerated from rest to a speed v, the object acquires kinetic energy in addition to its rest energy. The total energy E is the sum of the rest energy E_0 and the kinetic energy KE, or $E = E_0 + \text{KE}$. Therefore, the kinetic energy is the difference between the object's total energy and its rest energy. Using Equations 28.4 and 28.5, we can write the kinetic energy as

$$\mathrm{KE} = E - E_0 = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) \tag{28.6}$$

This equation is the relativistically correct expression for the kinetic energy of an object of mass m moving at speed v.

Equation 28.6 looks nothing like the kinetic energy expression introduced in Chapter 6—namely, $\mathrm{KE} = \frac{1}{2}mv^2$. However, for speeds much less than the speed of light ($v \ll c$), the relativistic equation for the kinetic energy reduces to $\mathrm{KE} = \frac{1}{2}mv^2$, as can be seen by using the binomial expansion* to represent the square root term in Equation 28.6:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots$$

Suppose v is much smaller than c—say $v = 0.01c$. The second term in the expansion has the value $\frac{1}{2}(v^2/c^2) = 5.0 \times 10^{-5}$, while the third term has the much smaller value $\frac{3}{8}(v^2/c^2)^2 = 3.8 \times 10^{-9}$. The additional terms are smaller still, so if $v \ll c$, we can neglect the third and additional terms in comparison with the first and second terms. Substituting the first two terms into Equation 28.6 gives

$$\mathrm{KE} \approx mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \tfrac{1}{2}mv^2$$

which is the familiar form for the kinetic energy. However, Equation 28.6 gives the correct kinetic energy for all speeds and must be used for speeds near the speed of light, as in Example 7.

Example 7 A High-Speed Electron

An electron ($m = 9.109 \times 10^{-31}$ kg) is accelerated from rest to a speed of $v = 0.9995c$ in a particle accelerator. Determine the electron's (a) rest energy, (b) total energy, and (c) kinetic energy in millions of electron volts or MeV.

Reasoning and Solution

(a) The electron's rest energy is

$$E_0 = mc^2 = (9.109 \times 10^{-31}\ \mathrm{kg})(2.998 \times 10^8\ \mathrm{m/s})^2 = 8.187 \times 10^{-14}\ \mathrm{J} \tag{28.5}$$

Since 1 eV = 1.602×10^{-19} J, the electron's rest energy is

$$(8.187 \times 10^{-14}\ \mathrm{J})\left(\frac{1\ \mathrm{eV}}{1.602 \times 10^{-19}\ \mathrm{J}}\right) = \boxed{5.11 \times 10^5\ \mathrm{eV}\quad \text{or}\quad 0.511\ \mathrm{MeV}}$$

(b) The total energy of an electron traveling at a speed of $v = 0.9995c$ is

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{(9.109 \times 10^{-31}\ \mathrm{kg})(2.998 \times 10^8\ \mathrm{m/s})^2}{\sqrt{1-\left(\frac{0.9995c}{c}\right)^2}} \tag{28.4}$$

$$= \boxed{2.59 \times 10^{-12}\ \mathrm{J}\quad \text{or}\quad 16.2\ \mathrm{MeV}}$$

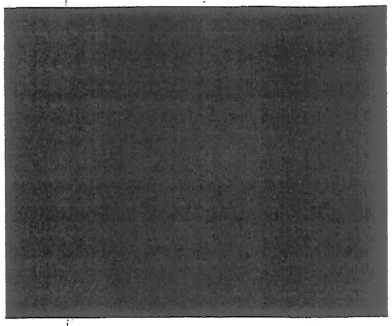
(c) The kinetic energy is the difference between the total energy and the rest energy:

$$\mathrm{KE} = E - E_0 = 2.59 \times 10^{-12}\ \mathrm{J} - 8.2 \times 10^{-14}\ \mathrm{J} \tag{28.6}$$

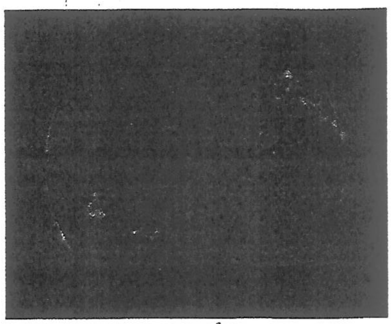
$$= \boxed{2.51 \times 10^{-12}\ \mathrm{J}\quad \text{or}\quad 15.7\ \mathrm{MeV}}$$

For comparison, if the kinetic energy of the electron had been calculated from $\frac{1}{2}mv^2$, a value of only 0.26 MeV would have been obtained.

* The binomial expansion states that $(1 - x)^n = 1 - nx + n(n - 1)x^2/2 + \cdots$. In our case, $x = v^2/c^2$ and $n = -1/2$.



Visible light image.



X-ray image.

Figure 28.9 The sun emits electromagnetic energy over a broad portion of the electromagnetic spectrum. These photographs were obtained using that energy in the indicated regions of the spectrum. (*Top photo:* © Mark Marten/NASA/Photo Researchers; *bottom photo:* Dr. Leon Golub/Photo Researchers)

Since mass and energy are equivalent, any change in one is accompanied by a corresponding change in the other. For instance, life on earth is dependent on electromagnetic energy (light) from the sun. Because this energy is leaving the sun (see Figure 28.9), there is a decrease in the sun's mass. Example 8 illustrates how to determine this decrease.

Example 8 The Sun Is Losing Mass

The sun radiates electromagnetic energy at the rate of 3.92×10^{26} W. (a) What is the change in the sun's mass during each second that it is radiating energy? (b) The mass of the sun is 1.99×10^{30} kg. What fraction of the sun's mass is lost during a human lifetime of 75 years?

Reasoning Since 1 W = 1 J/s, the amount of electromagnetic energy radiated during each second is 3.92×10^{26} J. Thus, during each second, the sun's rest energy decreases by this amount. The change ΔE_0 in the sun's rest energy is related to the change Δm in its mass by $\Delta E_0 = (\Delta m)c^2$, according to Equation 28.5.

Solution

(a) For each second that the sun radiates energy, the change in its mass is

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{3.92 \times 10^{26}\ \text{J}}{(3.00 \times 10^8\ \text{m/s})^2} = \boxed{4.36 \times 10^9\ \text{kg}}$$

Over 4 billion kilograms of mass are lost by the sun during each second.

(b) The amount of mass lost by the sun in 75 years is

$$\Delta m = (4.36 \times 10^9\ \text{kg/s})\left(\frac{3.16 \times 10^7\ \text{s}}{1\ \text{year}}\right)(75\ \text{years}) = 1.0 \times 10^{19}\ \text{kg}$$

Although this is an enormous amount of mass, it represents only a tiny fraction of the sun's total mass:

$$\frac{\Delta m}{m_{\text{sun}}} = \frac{1.0 \times 10^{19}\ \text{kg}}{1.99 \times 10^{30}\ \text{kg}} = \boxed{5.0 \times 10^{-12}}$$

Any change in the energy of a system causes a change in the mass of the system according to $\Delta E_0 = (\Delta m)c^2$. It does not matter whether the change in energy is due to a change in electromagnetic energy, potential energy, thermal energy, or so on. Although any change in energy gives rise to a change in mass, in most instances the change in mass is too small to be detected. For instance, when 4186 J of heat is used to raise the temperature of 1 kg of water by 1 C°, the mass changes by only $\Delta m = \Delta E_0/c^2 = (4186\ \text{J})/(3.00 \times 10^8\ \text{m/s})^2 = 4.7 \times 10^{-14}$ kg. Conceptual Example 9 illustrates further how a change in the energy of an object leads to an equivalent change in its mass.

28.7 The Relativistic Addition of Velocities

The velocity of an object relative to an observer plays a central role in special relativity, and to determine this velocity, it is sometimes necessary to add two or more velocities together. We first encountered relative velocity in Section 3.4, so we will begin by reviewing some of the ideas presented there. Figure 28.11 illustrates a truck moving at a constant velocity of $v_{TG} = +15$ m/s relative to an observer standing on the ground, where the plus sign denotes a direction to the right. Suppose someone on the truck throws a baseball toward the observer at a velocity of $v_{BT} = +8.0$ m/s relative to the truck. We might conclude that the observer on the ground sees the ball approaching at a velocity of $v_{BG} = v_{BT} + v_{TG} = 8.0\text{ m/s} + 15\text{ m/s} = +23\text{ m/s}$. These symbols are similar to those used in Section 3.4 and have the following meaning:

$v_{\boxed{BG}}$ = velocity of the **Baseball** relative to the **Ground** = +23 m/s

$v_{\boxed{BT}}$ = velocity of the **Baseball** relative to the **Truck** = +8.0 m/s

$v_{\boxed{TG}}$ = velocity of the **Truck** relative to the **Ground** = +15.0 m/s

Although the result that $v_{BG} = +23$m/s seems reasonable, careful measurements would show that it is not quite right. According to special relativity, the equation $v_{BG} = v_{BT} + v_{TG}$ is not valid for the following reason. If the velocity of the truck had a magnitude sufficiently close to the speed of light, the equation would predict that the observer on the earth could see the baseball moving faster than the speed of light. This is not possible, since no object with a finite mass can move faster than the speed of light.

For the case where the truck and ball are moving along the same straight line, the theory of special relativity reveals that the velocities are related according to

$$v_{BG} = \frac{v_{BT} + v_{TG}}{1 + \dfrac{v_{BT}v_{TG}}{c^2}}$$

The subscripts in this equation have been chosen for the specific situation shown in Figure 28.11. For the general situation, the relative velocities are related by the ***velocity-addition formula:***

Velocity addition

$$v_{AB} = \frac{v_{AC} + v_{CB}}{1 + \dfrac{v_{AC}v_{CB}}{c^2}} \tag{28.8}$$

where all the velocities are assumed to be constant and the symbols have the following meanings:

$v_{\boxed{AB}}$ = velocity of **object A** relative to **object B**

$v_{\boxed{AC}}$ = velocity of **object A** relative to **object C**

$v_{\boxed{CB}}$ = velocity of **object C** relative to **object B**

The ordering of the subscripts in Equation 28.8 follows the discussion in Section 3.4. For motion along a straight line, the velocities can have either positive or negative values, depending on whether they are directed along the positive or negative direction. Furthermore, switching the order of the subscripts changes the sign of the velocity, so, for example, $v_{BA} = -v_{AB}$ (see Example 11 in Chapter 3).

Equation 28.8 differs from the nonrelativistic formula ($v_{AB} = v_{AC} + v_{CB}$) by the presence of the $v_{AC}v_{CB}/c^2$ term in the denominator. This term arises because of the effects

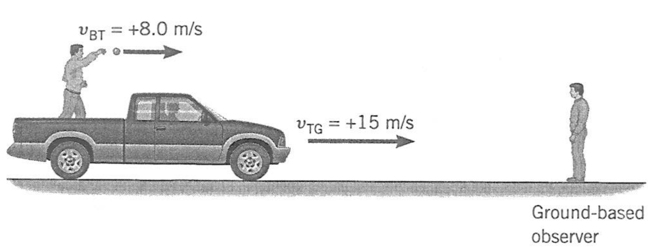


Figure 28.11 The truck is approaching the ground-based observer at a relative velocity of $v_{TG} = +15$ m/s. The velocity of the baseball relative to the truck is $v_{BT} = +8.0$ m/s.

of time dilation and length contraction that occur in special relativity. When v_{AC} and v_{CB} are small compared to c, the $v_{AC}v_{CB}/c^2$ term is small compared to 1, so the velocity-addition formula reduces to $v_{AB} \approx v_{AC} + v_{CB}$. However, when either v_{AC} or v_{CB} is comparable to c, the results can be quite different, as Example 10 illustrates.

Example 10 The Relativistic Addition of Velocities

Imagine a hypothetical situation in which the truck in Figure 28.11 is moving relative to the ground with a velocity of $v_{TG} = +0.8c$. A person riding on the truck throws a baseball at a velocity relative to the truck of $v_{BT} = +0.5c$. What is the velocity v_{BG} of the baseball relative to a person standing on the ground?

Reasoning The observer on the ground does *not* see the baseball approaching at $v_{BG} = 0.5c + 0.8c = 1.3c$. This cannot be because the speed of the ball would then exceed the speed of light. The velocity-addition formula gives the correct velocity, which has a magnitude less than the speed of light.

Solution The ground-based observer sees the ball approaching with a velocity of

$$v_{BG} = \frac{v_{BT} + v_{TG}}{1 + \dfrac{v_{BT}v_{TG}}{c^2}} = \frac{0.5c + 0.8c}{1 + \dfrac{(0.5c)(0.8c)}{c^2}} = \boxed{0.93c} \qquad (28.8)$$

Example 10 discusses how the speed of a baseball is viewed by observers in different inertial reference frames. The next example deals with a similar situation, except that the baseball is replaced by the light of a laser beam.

Conceptual Example 11 The Speed of a Laser Beam

Figure 28.12 shows an intergalactic cruiser approaching a hostile spacecraft. The velocity of the cruiser relative to the spacecraft is $v_{CS} = +0.7c$. Both vehicles are moving at a constant velocity. The cruiser fires a beam of laser light at the enemy. The velocity of the laser beam relative to the cruiser is $v_{LC} = +c$. (a) What is the velocity of the laser beam v_{LS} relative to the renegades aboard the spacecraft? (b) At what velocity do the renegades aboard the spacecraft see the laser beam move away from the cruiser?

Reasoning and Solution

(a) Since both vehicles move at a constant velocity, each constitutes an inertial reference frame. According to the speed of light postulate, *all* observers in inertial reference frames measure the speed of light in a vacuum to be c. Thus, the renegades aboard the hostile spacecraft see the laser beam travel toward them at the speed of light, even though the beam is emitted from the cruiser, which itself is moving at seven-tenths the speed of light.

(b) The renegades aboard the spacecraft see the cruiser approach them at a relative velocity of $v_{CS} = +0.7c$, and they also see the laser beam approach them at a relative velocity of $v_{LS} = +c$. Both these velocities are measured relative to the *same* inertial reference frame—namely, that of the spacecraft. Therefore, the renegades aboard the spacecraft see the laser beam move away from the cruiser at a velocity that is the difference between these two velocities, or $+c - (+0.7c) = +0.3c$. The velocity-addition formula, Equation 28.8, is not applicable here because both velocities are measured relative to the *same* inertial reference frame (the spacecraft's reference frame). The velocity-addition formula can be used only when the velocities are measured relative to different inertial reference frames.

Related Homework: *Conceptual Question 12, Problem 34*

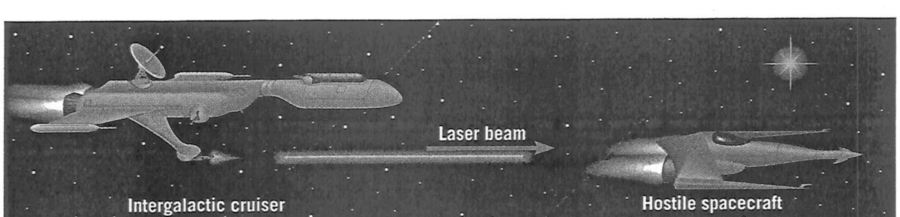


Figure 28.12 An intergalactic cruiser, closing in on a hostile spacecraft, fires a beam of laser light.

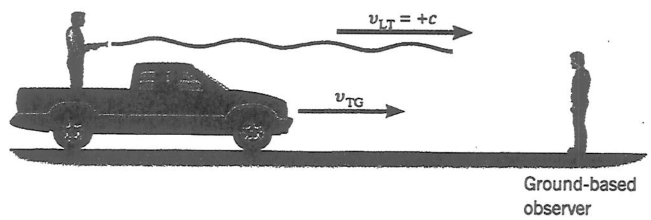


Figure 28.13 The speed of the light emitted by the flashlight is c relative to both the truck and the observer on the ground.

It is a straightforward matter to show that the velocity-addition formula is consistent with the speed of light postulate. Consider Figure 28.13, which shows a person riding on a truck and holding a flashlight. The velocity of the light, relative to the person on the truck, is $v_{LT} = +c$. The velocity v_{LG} of the light relative to the observer standing on the ground is given by the velocity-addition formula as

$$v_{LG} = \frac{v_{LT} + v_{TG}}{1 + \frac{v_{LT}v_{TG}}{c^2}} = \frac{c + v_{TG}}{1 + \frac{cv_{TG}}{c^2}} = \frac{(c + v_{TG})c}{(c + v_{TG})} = c$$

Thus, the velocity-addition formula indicates that the observer on the ground and the person on the truck both measure the speed of light to be c, independent of the relative velocity v_{TG} between them. This is exactly what the speed of light postulate states.