

Relativity Practice Problems

Time

According to Martians, Mars rotates about its axis once every 88 642 s. A spacecraft heads directly toward Mars from Earth at a speed of $0.800c$. What is the rotational period of Mars according to the humans on the spaceship?

[148 000 s]

A missile carries a timer designed to explode the device 2.00 s after it is armed. The missile is launched from earth and accelerated to an unknown final speed. After reaching its final speed, however, the bomb is observed by people on earth to explode 4.25 s after it is armed. What is the final speed of the bomb just before it explodes?

[0.882c]

Energy

A muon particle has rest energy 105 MeV (energy unit $1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}$). What is its kinetic energy (total energy minus rest energy) when its speed is $0.95c$?

[231 MeV]

The average power output of a nuclear power plant is 500 MW. In one minute, what is the change in the mass of the nuclear fuel due to the energy being taken from the nuclear reactor?

(Assume 100% efficiency).

[3.3 x 10⁻⁷ kg]

but than for himself. We can see how this curious effect arises with the help of the clock illustrated in Figure 28.4, which uses a pulse of light to mark time. A short pulse of light is emitted by a light source, reflects from a mirror, and then strikes a detector that is situated next to the source. Each time a pulse reaches the detector, a “tick” registers on the chart recorder, another short pulse of light is emitted, and the cycle repeats. Thus, the time interval between successive “ticks” is marked by a beginning event (the firing of the light source) and an ending event (the pulse striking the detector). The source and detector are so close to each other that the two events can be considered to occur at the same location.

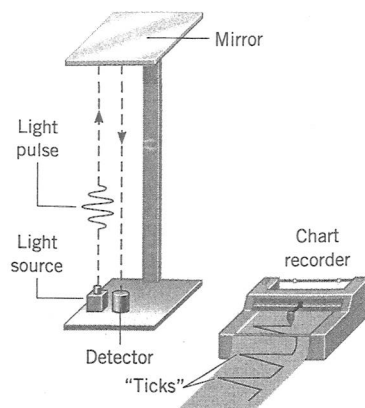


Figure 28.4 A light clock.

Suppose two identical clocks are built. One is kept on earth, and the other is placed aboard a spacecraft that travels at a constant velocity relative to the earth. The astronaut is at rest with respect to the clock on the spacecraft and, therefore, sees the light pulse move along the up/down path shown in Figure 28.5a. According to the astronaut, the time interval Δt_0 required for the light to follow this path is the distance $2D$ divided by the speed of light c ; $\Delta t_0 = 2D/c$. To the astronaut, Δt_0 is the time interval between the “ticks” of the spacecraft clock—that is, the time interval between the beginning and ending events of the clock. An earth-based observer, however, does *not* measure Δt_0 as the time interval between these two events. Since the spacecraft is moving, the earth-based observer sees the light pulse follow the diagonal path shown in red in part *b* of the drawing. This path is longer than the up/down path seen by the astronaut. But light travels at the *same speed* c for both observers, in accord with the speed of light postulate. Therefore, the earth-based observer measures a time interval Δt between the two events that is *greater* than the time interval Δt_0 measured by the astronaut. In other words, the earth-based observer, using his own earth-based clock to measure the performance of the astronaut’s clock, finds that the astronaut’s clock runs slowly. This result of special relativity is known as *time dilation*. (To *dilate* means to expand, and the time interval Δt is “expanded” relative to Δt_0 .)

The time interval Δt that the earth-based observer measures in Figure 28.5b can be determined as follows. While the light pulse travels from the source to the detector, the spacecraft moves a distance $2L = v\Delta t$ to the right, where v is the speed of the spacecraft relative to the earth. From the drawing it can be seen that the light pulse travels a total diagonal distance of $2s$ during the time interval Δt . Applying the Pythagorean theorem, we find that

$$2s = 2\sqrt{D^2 + L^2} = 2\sqrt{D^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

But the distance $2s$ is also equal to the speed of light times the time interval Δt , so $2s = c\Delta t$. Therefore,

$$c\Delta t = 2\sqrt{D^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

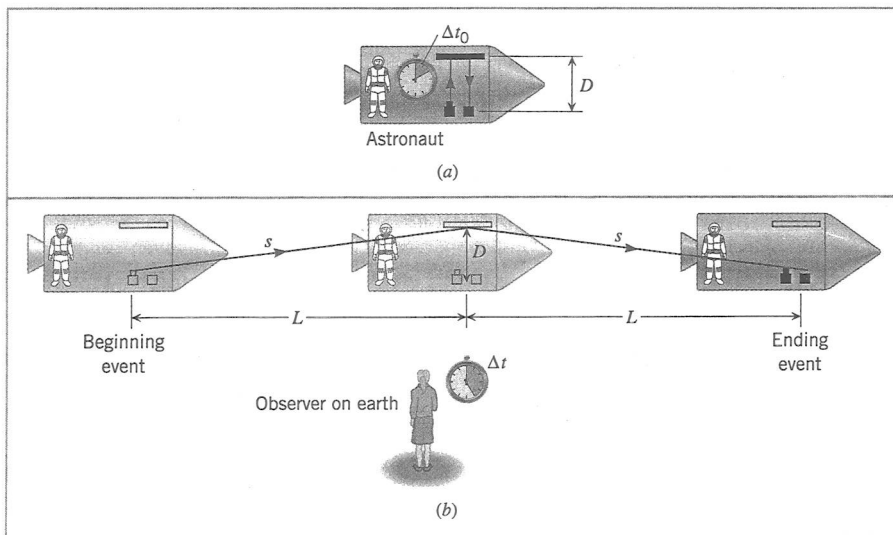


Figure 28.5 (a) The astronaut measures the time interval Δt_0 between successive “ticks” of his light clock. (b) An observer on earth watches the astronaut’s clock and sees the light pulse travel a greater distance between “ticks” than it does in part *a*. Consequently, the earth-based observer measures a time interval Δt between “ticks” that is greater than Δt_0 .

Squaring this result and solving for Δt gives

$$\Delta t = \frac{2D}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

But $2D/c = \Delta t_0$, the time interval between successive “ticks” of the spacecraft’s clock as measured by the astronaut. With this substitution, the equation for Δt can be expressed as

Time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (28.1)$$

The symbols in this formula are defined as follows:

Δt_0 = proper time interval, which is the interval between two events as measured by an observer who is at rest with respect to the events and who views them as occurring *at the same place*

Δt = dilated time interval, which is the interval measured by an observer who is in motion with respect to the events and who views them as occurring at *different places*

v = relative speed between the two observers

c = speed of light in a vacuum

For a speed v that is less than c , the term $\sqrt{1 - v^2/c^2}$ in Equation 28.1 is less than 1, and the dilated time interval Δt is greater than Δt_0 . Example 1 shows this time dilation effect.

Example 1 Time Dilation

The spacecraft in Figure 28.5 is moving past the earth at a constant speed v that is 0.92 times the speed of light. Thus, $v = (0.92)(3.0 \times 10^8 \text{ m/s})$, which is often written as $v = 0.92c$. The astronaut measures the time interval between successive “ticks” of the spacecraft clock to be $\Delta t_0 = 1.0 \text{ s}$. What is the time interval Δt that an earth observer measures between “ticks” of the astronaut’s clock?

Reasoning Since the clock on the spacecraft is moving relative to the earth observer, the earth observer measures a greater time interval Δt between “ticks” than does the astronaut, who is at rest relative to the clock. The dilated time interval Δt can be determined from the time dilation relation, Equation 28.1.

Solution The dilated time interval is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.0 \text{ s}}{\sqrt{1 - \left(\frac{0.92c}{c}\right)^2}} = \boxed{2.6 \text{ s}}$$

From the point of view of the earth-based observer, the astronaut is using a clock that is running slowly, because the earth-based observer measures a time between “ticks” that is longer (2.6 s) than what the astronaut measures (1.0 s).

Present-day spacecrafts fly nowhere near as fast as the craft in Example 1. Yet circumstances exist in which time dilation can create appreciable errors if not accounted for. The Global Positioning System (GPS), for instance, uses highly accurate and stable atomic clocks on board each of 24 satellites orbiting the earth at speeds of about 4000 m/s. These clocks make it possible to measure the time it takes for electromagnetic waves to travel from a satellite to a ground-based GPS receiver. From the speed of light and the times measured for signals from three or more of the satellites, it is possible to locate the position of the receiver (see Section 5.5). The stability of the clocks must be better than one part in 10^{13} to ensure the positional accuracy demanded of the GPS. Using Equation 28.1 and the speed of the GPS satellites, we can calculate the difference between the dilated time interval and the proper time interval as a fraction of the proper time interval and compare the result to the stability of the GPS clocks:

$$\begin{aligned} \frac{\Delta t - \Delta t_0}{\Delta t_0} &= \frac{1}{\sqrt{1 - v^2/c^2}} - 1 = \frac{1}{\sqrt{1 - (4000 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}} - 1 \\ &= \frac{1}{1.1 \times 10^{10}} \end{aligned}$$

This result is approximately one thousand times greater than the GPS-clock stability of one part in 10^{13} . Thus, if not taken into account, time dilation would cause an error in the measured position of the earth-based GPS receiver roughly equivalent to that caused by a thousand-fold degradation in the stability of the atomic clocks.

PROPER TIME INTERVAL

In Figure 28.5 both the astronaut and the person standing on the earth are measuring the time interval between a beginning event (the firing of the light source) and an ending event (the light pulse striking the detector). For the astronaut, who is at rest with respect to the light clock, the two events occur at the same location. (Remember, we are assuming that the light source and detector are so close together that they are considered to be at the same place.) Being at rest with respect to a clock is the usual or “proper” situation, so the time interval Δt_0 measured by the astronaut is called the *proper time interval*. In general, the proper time interval Δt_0 between two events is the time interval measured by an observer who is at rest relative to the events and sees them at the *same location* in space. On the other hand, the earth-based observer does not see the two events occurring at the same location in space, since the spacecraft is in motion. The time interval Δt that the earth-based observer measures is, therefore, not a proper time interval in the sense that we have defined it.

To understand situations involving time dilation, it is essential to distinguish between Δt_0 and Δt . It is helpful if one first identifies the two events that define the time interval. These may be something other than the firing of a light source and the light pulse striking a detector. Then determine the reference frame in which the two events occur at the same place. An observer at rest in this reference frame measures the proper time interval Δt_0 .

SPACE TRAVEL

One of the intriguing aspects of time dilation occurs in conjunction with space travel. Since enormous distances are involved, travel to even the closest star outside our solar system would take a long time. However, as the following example shows, the travel time can be considerably less for the passengers than one might guess.

Example 2 Space Travel

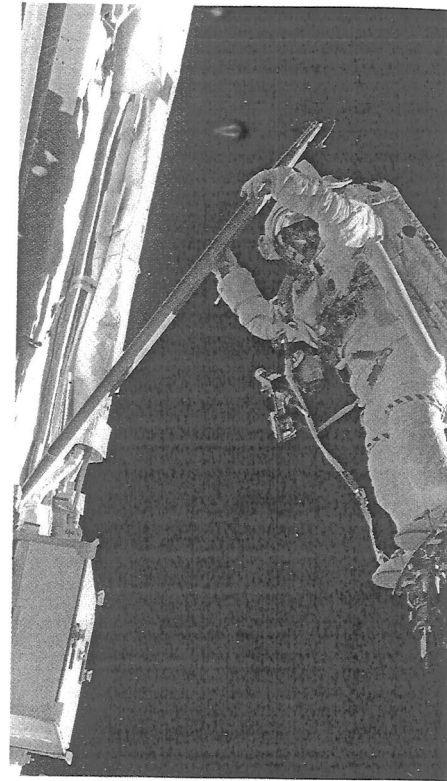
Alpha Centauri, a nearby star in our galaxy, is 4.3 light-years away. This means that, as measured by a person on earth, it would take light 4.3 years to reach this star. If a rocket leaves for Alpha Centauri and travels at a speed of $v = 0.95c$ relative to the earth, by how much will the passengers have aged, according to their own clock, when they reach their destination? Assume that the earth and Alpha Centauri are stationary with respect to one another.

Reasoning The two events in this problem are the departure from earth and the arrival at Alpha Centauri. At departure, earth is just outside the spaceship. Upon arrival at the destination, Alpha Centauri is just outside. Therefore, relative to the passengers, the two events occur at the same place—namely, just outside the spaceship. Thus, the passengers measure the proper time interval Δt_0 on their clock, and it is this interval that we must find. For a person left behind on earth, the events occur at *different places*, so such a person measures the dilated time interval Δt rather than the proper time interval. To find Δt we note that the time to travel a given distance is inversely proportional to the speed. Since it takes 4.3 years to traverse the distance between earth and Alpha Centauri at the speed of light, it would take even longer at the slower speed of $v = 0.95c$. Thus, a person on earth measures the dilated time interval to be $\Delta t = (4.3 \text{ years})/0.95 = 4.5 \text{ years}$. This value can be used with the time-dilation equation to find the proper time interval Δt_0 .

Solution Using the time-dilation equation, we find that the proper time interval by which the passengers judge their own aging is

$$\Delta t_0 = \Delta t \sqrt{1 - \frac{v^2}{c^2}} = (4.5 \text{ years}) \sqrt{1 - \left(\frac{0.95c}{c}\right)^2} = \boxed{1.4 \text{ years}}$$

Thus, the people aboard the rocket will have aged by only 1.4 years when they reach Alpha Centauri, and not the 4.5 years an earthbound observer has calculated.



Shown here in orbit is astronaut Lee M. E. Morin on April 13, 2002, as he works on the International Space Station. His feet are secured to the end of the station's robotic arm. (© AP/Wide World Photos)

The physics of space travel and special relativity.

Problem solving insight
In dealing with time dilation, decide which interval is the proper time interval as follows: (1) Identify the two events that define the interval. (2) Determine the reference frame in which the events occur at the same place; an observer at rest in this frame measures the proper time interval Δt_0 .

VERIFICATION OF TIME DILATION

A striking confirmation of time dilation was achieved in 1971 by an experiment carried out by J. C. Hafele and R. E. Keating.* They transported very precise cesium-beam atomic clocks around the world on commercial jets. Since the speed of a jet plane is considerably less than c , the time-dilation effect is extremely small. However, the atomic clocks were accurate to about $\pm 10^{-9}$ s, so the effect could be measured. The clocks were in the air for 45 hours, and their times were compared to reference atomic clocks kept on earth. The experimental results revealed that, within experimental error, the readings on the clocks on the planes were different from those on earth by an amount that agreed with the prediction of relativity.

The behavior of subatomic particles called *muons* provides additional confirmation of time dilation. These particles are created high in the atmosphere, at altitudes of about 10 000 m. When at rest, muons exist only for about 2.2×10^{-6} s before disintegrating. With such a short lifetime, these particles could never make it down to the earth's surface, even traveling at nearly the speed of light. However, *a large number of muons do reach the earth*. The only way they can do so is to live longer because of time dilation, as Example 3 illustrates.

Example 3 The Lifetime of a Muon

The average lifetime of a muon at rest is 2.2×10^{-6} s. A muon created in the upper atmosphere, thousands of meters above sea level, travels toward the earth at a speed of $v = 0.998c$. Find, on the average, (a) how long a muon lives according to an observer on earth, and (b) how far the muon travels before disintegrating.

Reasoning The two events of interest are the generation and subsequent disintegration of the muon. When the muon is at rest, these events occur at the same place, so the muon's average (at rest) lifetime of 2.2×10^{-6} s is a proper time interval Δt_0 . When the muon moves at a speed $v = 0.998c$ relative to the earth, an observer on the earth measures a dilated lifetime Δt that is given by Equation 28.1. The average distance x traveled by a muon, as measured by an earth observer, is equal to the muon's speed times the dilated time interval.

Solution

(a) The observer on earth measures a dilated lifetime given by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6} \text{ s}}{\sqrt{1 - \left(\frac{0.998c}{c}\right)^2}} = \boxed{35 \times 10^{-6} \text{ s}} \quad (28.1)$$

(b) The distance traveled by the muon before it disintegrates is

$$x = v \Delta t = (0.998)(3.00 \times 10^8 \text{ m/s})(35 \times 10^{-6} \text{ s}) = \boxed{1.0 \times 10^4 \text{ m}}$$

Thus, the dilated, or extended, lifetime provides sufficient time for the muon to reach the surface of the earth. If its lifetime were only 2.2×10^{-6} s, a muon would travel only 660 m before disintegrating and could never reach the earth.

Problem solving insight

The proper time interval Δt_0 is always shorter than the dilated time interval Δt .

28
Be
me
vio
tak
the
are
ear
bot
two
tim
if th
mir
yea
 $v\Delta$
also
one

con
the
tim
 $v =$
the
pas
spe
con
sult

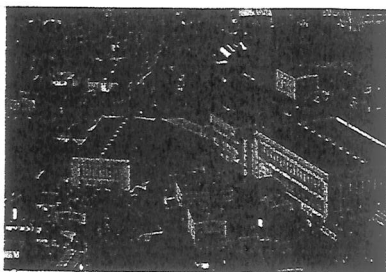


Figure 28.8 The Stanford three-kilometer linear accelerator accelerates electrons almost to the speed of light.
(© Bill Marsh/Photo Researchers)

28.6 The Equivalence of Mass and Energy

THE TOTAL ENERGY OF AN OBJECT

One of the most astonishing results of special relativity is that mass and energy are equivalent, in the sense that a gain or loss of mass can be regarded equally well as a gain or loss of energy. Consider, for example, an object of mass m traveling at a speed v . Einstein showed that the **total energy** E of the moving object is related to its mass and speed by the following relation:

$$\begin{array}{l} \text{Total energy} \\ \text{of an object} \end{array} \quad E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (28.4)$$

To gain some understanding of Equation 28.4, consider the special case in which the object is at rest. When $v = 0$ m/s, the total energy is called the **rest energy** E_0 , and Equation 28.4 reduces to Einstein's now-famous equation:

$$\begin{array}{l} \text{Rest energy} \\ \text{of an object} \end{array} \quad E_0 = mc^2 \quad (28.5)$$

The rest energy represents the energy equivalent of the mass of an object at rest. As Example 6 shows, even a small mass is equivalent to an enormous amount of energy.

Example 6 The Energy Equivalent of a Golf Ball

A 0.046-kg golf ball is lying on the green. (a) Find the rest energy of the golf ball. (b) If this rest energy were used to operate a 75-W light bulb, for how many years could the bulb stay on?

Reasoning The rest energy E_0 that is equivalent to the mass m of the golf ball is found from the relation $E_0 = mc^2$. The power used by the bulb is 75 W, which means that it consumes 75 J of energy per second. If the entire rest energy of the ball were available for use, the bulb could stay on for a time equal to the rest energy divided by the power.

Solution

(a) The rest energy of the ball is

$$E_0 = mc^2 = (0.046 \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 = \boxed{4.1 \times 10^{15} \text{ J}} \quad (28.5)$$

(b) This rest energy can keep the bulb burning for a time t given by

$$t = \frac{\text{Rest energy}}{\text{Power}} = \frac{4.1 \times 10^{15} \text{ J}}{75 \text{ W}} = 5.5 \times 10^{13} \text{ s} \quad (6.10b)$$

Since one year contains 3.2×10^7 s, we find $t = \boxed{1.7 \times 10^6 \text{ yr}}$, or 1.7 million years!

When an object is accelerated from rest to a speed v , the object acquires kinetic energy in addition to its rest energy. The total energy E is the sum of the rest energy E_0 and the kinetic energy KE, or $E = E_0 + \text{KE}$. Therefore, the kinetic energy is the difference

between the object's total energy and its rest energy. Using Equations 28.4 and 28.5, we can write the kinetic energy as

$$\text{KE} = E - E_0 = mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \quad (28.6)$$

This equation is the relativistically correct expression for the kinetic energy of an object of mass m moving at speed v .

Equation 28.6 looks nothing like the kinetic energy expression introduced in Chapter 6—namely, $\text{KE} = \frac{1}{2}mv^2$. However, for speeds much less than the speed of light ($v \ll c$), the relativistic equation for the kinetic energy reduces to $\text{KE} = \frac{1}{2}mv^2$, as can be seen by using the binomial expansion* to represent the square root term in Equation 28.6:

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{1}{2} \left(\frac{v^2}{c^2} \right) + \frac{3}{8} \left(\frac{v^2}{c^2} \right)^2 + \dots$$

Suppose v is much smaller than c —say $v = 0.01c$. The second term in the expansion has the value $\frac{1}{2}(v^2/c^2) = 5.0 \times 10^{-5}$, while the third term has the much smaller value $\frac{3}{8}(v^2/c^2)^2 = 3.8 \times 10^{-9}$. The additional terms are smaller still, so if $v \ll c$, we can neglect the third and additional terms in comparison with the first and second terms. Substituting the first two terms into Equation 28.6 gives

$$\text{KE} \approx mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} - 1 \right) = \frac{1}{2}mv^2$$

which is the familiar form for the kinetic energy. However, Equation 28.6 gives the correct kinetic energy for all speeds and must be used for speeds near the speed of light, as in Example 7.

Example 7 A High-Speed Electron

An electron ($m = 9.109 \times 10^{-31}$ kg) is accelerated from rest to a speed of $v = 0.9995c$ in a particle accelerator. Determine the electron's (a) rest energy, (b) total energy, and (c) kinetic energy in millions of electron volts or MeV.

Reasoning and Solution

(a) The electron's rest energy is

$$E_0 = mc^2 = (9.109 \times 10^{-31} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 = 8.187 \times 10^{-14} \text{ J} \quad (28.5)$$

Since $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$, the electron's rest energy is

$$(8.187 \times 10^{-14} \text{ J}) \left(\frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J}} \right) = \boxed{5.11 \times 10^5 \text{ eV} \text{ or } 0.511 \text{ MeV}}$$

(b) The total energy of an electron traveling at a speed of $v = 0.9995c$ is

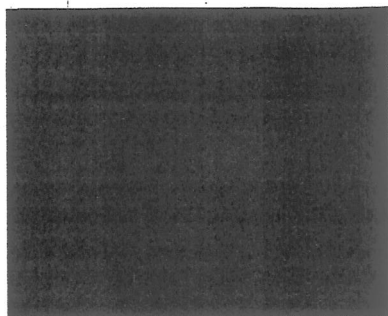
$$\begin{aligned} E &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{(9.109 \times 10^{-31} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2}{\sqrt{1 - \left(\frac{0.9995c}{c} \right)^2}} \quad (28.4) \\ &= \boxed{2.59 \times 10^{-12} \text{ J} \text{ or } 16.2 \text{ MeV}} \end{aligned}$$

(c) The kinetic energy is the difference between the total energy and the rest energy:

$$\begin{aligned} \text{KE} &= E - E_0 = 2.59 \times 10^{-12} \text{ J} - 8.2 \times 10^{-14} \text{ J} \quad (28.6) \\ &= \boxed{2.51 \times 10^{-12} \text{ J} \text{ or } 15.7 \text{ MeV}} \end{aligned}$$

For comparison, if the kinetic energy of the electron had been calculated from $\frac{1}{2}mv^2$, a value of only 0.26 MeV would have been obtained.

* The binomial expansion states that $(1 - x)^n = 1 - nx + n(n-1)x^2/2 + \dots$. In our case, $x = v^2/c^2$ and $n = -1/2$.



Visible light image.



X-ray image.

Figure 28.9 The sun emits electromagnetic energy over a broad portion of the electromagnetic spectrum. These photographs were obtained using that energy in the indicated regions of the spectrum. (Top photo: © Mark Marten/NASA/Photo Researchers; bottom photo: Dr. Leon Golub/Photo Researchers)

Since mass and energy are equivalent, any change in one is accompanied by a corresponding change in the other. For instance, life on earth is dependent on electromagnetic energy (light) from the sun. Because this energy is leaving the sun (see Figure 28.9), there is a decrease in the sun's mass. Example 8 illustrates how to determine this decrease.

Example 8 The Sun Is Losing Mass

The sun radiates electromagnetic energy at the rate of 3.92×10^{26} W. (a) What is the change in the sun's mass during each second that it is radiating energy? (b) The mass of the sun is 1.99×10^{30} kg. What fraction of the sun's mass is lost during a human lifetime of 75 years?

Reasoning Since $1 \text{ W} = 1 \text{ J/s}$, the amount of electromagnetic energy radiated during each second is 3.92×10^{26} J. Thus, during each second, the sun's rest energy decreases by this amount. The change ΔE_0 in the sun's rest energy is related to the change Δm in its mass by $\Delta E_0 = (\Delta m)c^2$, according to Equation 28.5.

Solution

(a) For each second that the sun radiates energy, the change in its mass is

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{3.92 \times 10^{26} \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = \boxed{4.36 \times 10^9 \text{ kg}}$$

Over 4 billion kilograms of mass are lost by the sun during each second.

(b) The amount of mass lost by the sun in 75 years is

$$\Delta m = (4.36 \times 10^9 \text{ kg/s}) \left(\frac{3.16 \times 10^7 \text{ s}}{1 \text{ year}} \right) (75 \text{ years}) = 1.0 \times 10^{19} \text{ kg}$$

Although this is an enormous amount of mass, it represents only a tiny fraction of the sun's total mass:

$$\frac{\Delta m}{m_{\text{sun}}} = \frac{1.0 \times 10^{19} \text{ kg}}{1.99 \times 10^{30} \text{ kg}} = \boxed{5.0 \times 10^{-12}}$$

Any change in the energy of a system causes a change in the mass of the system according to $\Delta E_0 = (\Delta m)c^2$. It does not matter whether the change in energy is due to a change in electromagnetic energy, potential energy, thermal energy, or so on. Although any change in energy gives rise to a change in mass, in most instances the change in mass is too small to be detected. For instance, when 4186 J of heat is used to raise the temperature of 1 kg of water by 1 C° , the mass changes by only $\Delta m = \Delta E_0/c^2 = (4186 \text{ J})/(3.00 \times 10^8 \text{ m/s})^2 = 4.7 \times 10^{-14} \text{ kg}$. Conceptual Example 9 illustrates further how a change in the energy of an object leads to an equivalent change in its mass.