Introduction Sheet 10

Calculus 2: Integration

Rules to Remember:

Integration is the reverse process of differentiation, taking one from dy/dx back to y. The **indefinite integral** of a function $f(x)$ with respect to x is a function $h(x)$ denoted

$$h(x) = \int dx\, f(x) \tag{1}$$

Its derivative $dh/dx = f(x)$. $f(x)$ is called the **integrand**.

Some simple special cases for integration:

- If $y = ax^n$, $n \neq -1$, then $\int dx\, y = ax^{n+1}/(n+1) + c$
- If $y = \sin x$, then $\int dx\, y = -\cos x + c$
- If $y = \cos x$, then $\int dx\, y = \sin x + c$
- If $y = e^x$, then $\int dx\, y = e^x + c$
- If $y = 1/x$, then $\int dx\, y = \ln x + c$

An arbitrary constant c is allowed because $dc/dx = 0$.

There many tricks for integrating more complicated functions (see Chap. 9 of *Maths: A Student Survival Guide*). They all boil down to manipulating the integrand into a form where you recognise it as the derivative of something. The simplest tricks are:

- $\int dx\, (af(x) + bg(x)) = a\int dx\, f(x) + b\int dx\, g(x)$ (linearity).
- Make a change of variable $x \to u(x)$ in the integrand, using $dx = du \times dx/du = du/(du/dx)$ to convert to an integral with respect to u (substitution):

$$\text{e.g. use } u = 2x+1 \text{ for } \int dx\, \frac{1}{(2x+1)} = \int \frac{du}{2}\frac{1}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln(2x+1) + c$$

- Use partial fractions to split the integrand into simpler parts:

$$\text{e.g. } \frac{1}{(4x^2-1)} = \frac{1}{2}\left(\frac{1}{(2x-1)} - \frac{1}{(2x+1)}\right)$$

Don't make the mistake of using differentiation rules (sheet 9) for integration, e.g. $\int dx\, f(x)g(x) \neq \int dx\, f(x) \times \int dx\, g(x)$.

If $h(x)$ is the indefinite integral, the **definite integral** of a function $f(x)$ between two limits $x = a$ and $x = b$ is denoted and defined by

$$\int_a^b dx\, f(x) = [h(x)]_a^b = h(b) - h(a) \tag{2}$$

It gives the (signed) area under the graph of $y = f(x)$ between $x = a$ and $x = b$.

Practice Questions:

P1 Integrate the following functions $x(t)$ with respect to the variable t, including a constant of integration:

a) $x(t) = 3t$ b) $x(t) = 4t^4$ c) $x(t) = \frac{\pi}{t^2}$
d) $x(t) = t^3(1+t)$ e) $x(t) = e^{2t+3}$ f) $x(t) = \cos 4t$

P2 Find the values of the following definite integrals (you need not evaluate 'e' or 'ln')

a) $\int_1^3 (t^2 + 2t + 1)\, dt$ b) $\int_0^{\pi/6} \cos x\, dx$ c) $\int_0^{\pi/4} 2\sin 2u\, du$
d) $\int_0^2 e^{2x}\, dx$ e) $\int_0^3 \frac{du}{2u+1}$ f) $\int_0^{\pi/4} \sec^2 x\, dx$

P3 Find the following indefinite integrals (don't forget to include a constant)

a) $\int \frac{x}{1+x^2}\, dx$ b) $\int \frac{1}{1-x^2}\, dx$ c) $\int \frac{3x}{(2x^2-3)^4}\, dx$
d) $\int \frac{6x^2}{\sqrt{2x^3+9}}\, dx$ e) $\int x\sqrt{1-x^2}\, dx$ f) $\int \frac{4}{(x+2)(x+3)}\, dx$

P4 A car's velocity dx/dt in the x-direction at time t is described by $dx/dt = 24\cos 2t \sin^2 2t$.

a) Give the expression for the displacement at time t relative to that at $t = 0$.

b) Give the distance travelled by the car after $\pi/12$s.

P5 If a spaceship accelerates, from rest at time $t = 0$, at a rate $2t/\sqrt{1+t^2}\text{ms}^{-2}$ at time t, calculate in years to 1 significant figure how long it would take to reach the speed of light $3 \times 10^8\text{ms}^{-1}$.

P6 In plan view, the shoreline of a bay may be approximated by a curve $f(x) = -x^2 + 2x$ between one headland at $x = 0$ and another at $x = 1$km, where f is also measured in km. At low tide, the edge of the sea just reaches each headland, following a straight line between them. Calculate in km^2 the area of the bay uncovered by water at low tide.