Introduction Sheet 9
Calculus 1: Differentiation

Rules to Remember:

For a function y of the variable x, dy/dx denotes the **derivative** of y with respect to x; its value is the rate of change of y with respect to x or the gradient of the graph of y versus x. The process of finding dy/dx is **differentiation**.

Some simple special cases for differentiation:

- If $y = ax^n$, then $dy/dx = nax^{n-1}$
- If $y = \sin x$, then $dy/dx = \cos x$
- If $y = \cos x$, then $dy/dx = -\sin x$
- If $y = e^x$, then $dy/dx = e^x$
- If $y = \ln x$, then $dy/dx = 1/x$

The above cases together with the following rules enable differentiation of more complicated functions:

- If $y = af(x) + bg(x)$, with a, b constant (independent of x), then $dy/dx = a\, df/dx + b\, dg/dx$ (linearity).
- If $y = f(x)g(x)$ then $dy/dx = f(x)dg/dx + g(x)df/dx$ (product rule).
- If $y = f(z)$ and $z = g(x)$ then $dy/dx = df/dz \times dg/dx$ (chain rule).
- $dx/dy = 1/(dy/dx)$ (reciprocity).

Since dy/dx is itself a function of x, it may also be differentiated using the above rules. This gives the 2nd derivative, d^2y/dx^2, whose value is the rate of change of the rate of change of y with respect to x or the curvature of the graph of y versus x.

If $dy/dx = 0$ for a particular value of x, the graph of y is flat there. It means the function is at a maximum, a minimum, or an inflection point. Which of these it is can be decided either by sketching the graph (preferable), or finding the curvature d^2y/dx^2 at that value of x (if < 0 then max, if > 0 then min).

Practice Questions:

P1 Differentiate the following functions $x(t)$ with respect to the variable t:

a) $x(t) = 3t$	b) $x(t) = 4t^4$	c) $x(t) = \frac{\pi}{t^2}$
d) $x(t) = t^3(1+t)$	e) $x(t) = \frac{1}{At^3+B}$	f) $x(t) = (A\sqrt{t}+B)^4$

P2 Find the rate of change of the functions $y(z)$ with respect to z

a) $y(z) = 2\sin z$	b) $y(z) = \cos^2 z$	c) $y(z) = e^{az}$
d) $y(z) = A\sin(e^{Bz})$	e) $y(z) = \tan z^3$	f) $y(z) = z\ln z^2$

P3 Using differentiation, find the *maximum* value of the following functions $f(x)$.

a) $f(x) = -x^2+x$	b) $f(x) = \ln x - x$	c) $f(x) = -x^4+2x^2$
d) $f(x) = \frac{x^2}{4}+\frac{4}{x}$	e) $f(x) = xe^{-2x^2}$	f) $f(x) = \frac{\sqrt{x-n}}{x}$; $n > 0$

[For e), leave your answer in terms of 'e']

P4 A car's motion in the x-direction at time t is described by $x(t) = At^2 + Bt$ where $A = 5\text{ms}^{-2}$ and $B = 10\text{ms}^{-1}$.

a) Give the expression at time t for
i) the velocity
ii) the acceleration

b) Give the car's position, velocity and acceleration after
i) 3 s
ii) 25 s

P5 The Nebraska Board of Grain are designing new portable grain silos. They have enough sheet material to make 2000 cylindrical containers, each of fixed surface area 54m^2 (this includes the cylinder ends). Calculate in terms of π the maximum volume of grain that could be stored in total.

P6 The height $h(x)$ in metres above the ground of a parachutist varies with her horizontal displacement x in meters from a landing target on the ground as $h(x) = 50\sin^{-1}(0.1x)$. What is the rate of change of h with respect to x at $x = 6$m?

[*Hint*: Differentiate $\sin(h/50)$ to get an equation involving dh/dx]