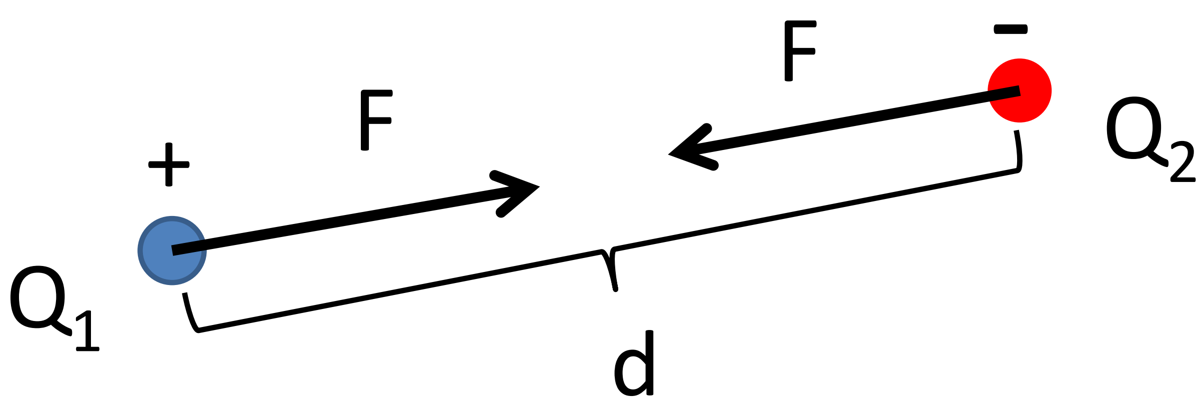
Gauss’s Law &
Parallel Plate Capacitors

Quick Electrostatics Refresher

Coulomb's Law: $|F| = \dfrac{1}{4\pi\varepsilon_0} \dfrac{|Q_1|\,|Q_2|}{d^2}$
(Electric Force)

Direction: opposites attract, likes repel

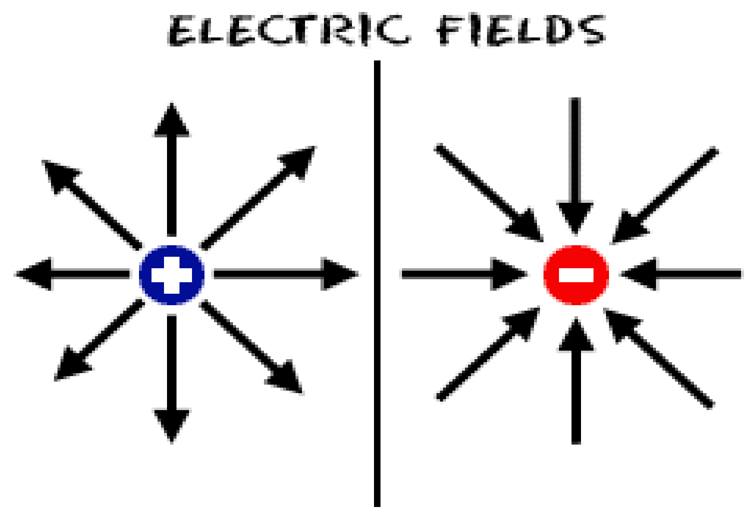


Alternative Approach: Every charge is a source of an electric field, E. Other charges interact with E.

source

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{d^2}$$

distance from Q



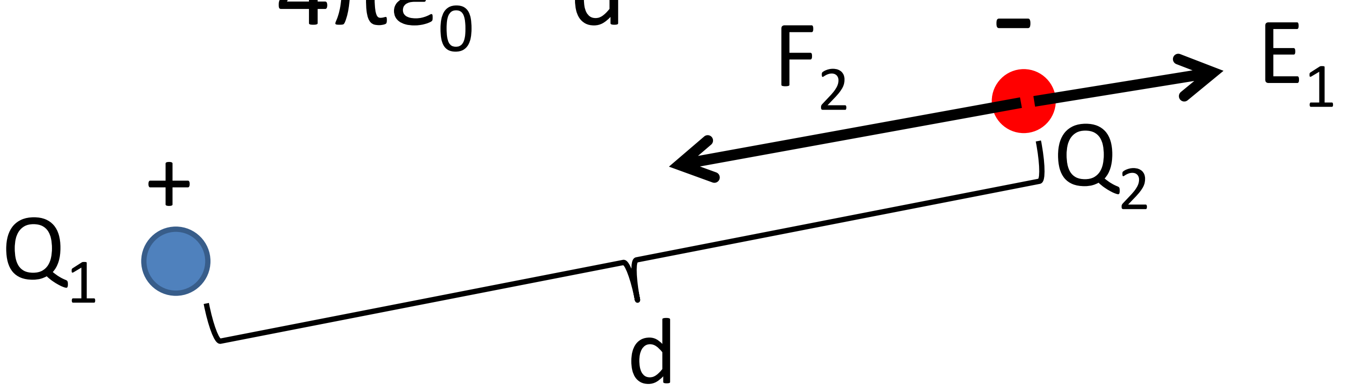
$F = qE$ (F on q in field E)

KEY: E is the force per charge

Example: Electric Force on Q_2

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{d^2}$$ (due to Q_1 at location of Q_2)

$$F_2 = Q_2 E_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{d^2}$$



Note: F on a + charge is in direction of E

F on a - charge is in the opposite direction

What is Gauss's Law?

- Fundamental relationship between charge and **flux** of the electric field through a surface enclosing the charge

- Equivalent to Coulomb's Law

What is electric flux (Φ)?

Mathematically: $$\boldsymbol{\Phi = \oint E \bullet da}$$

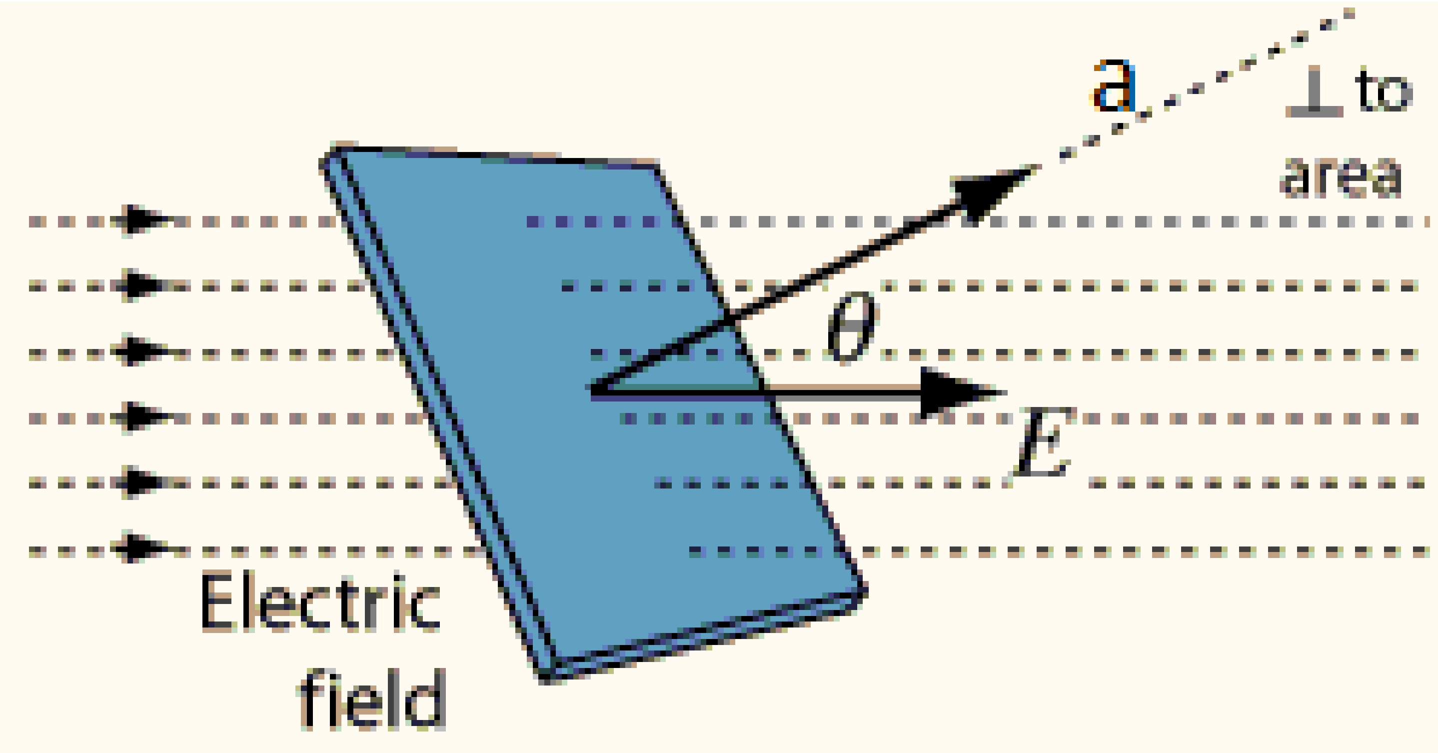
Sum over a surface of the product of the component of the electric field perpendicular to the surface and the area it penetrates

Conceptually:

Amount of electric field penetrating a surface

(Analogy: catching rain through a hoop

Flux Through Single Surface

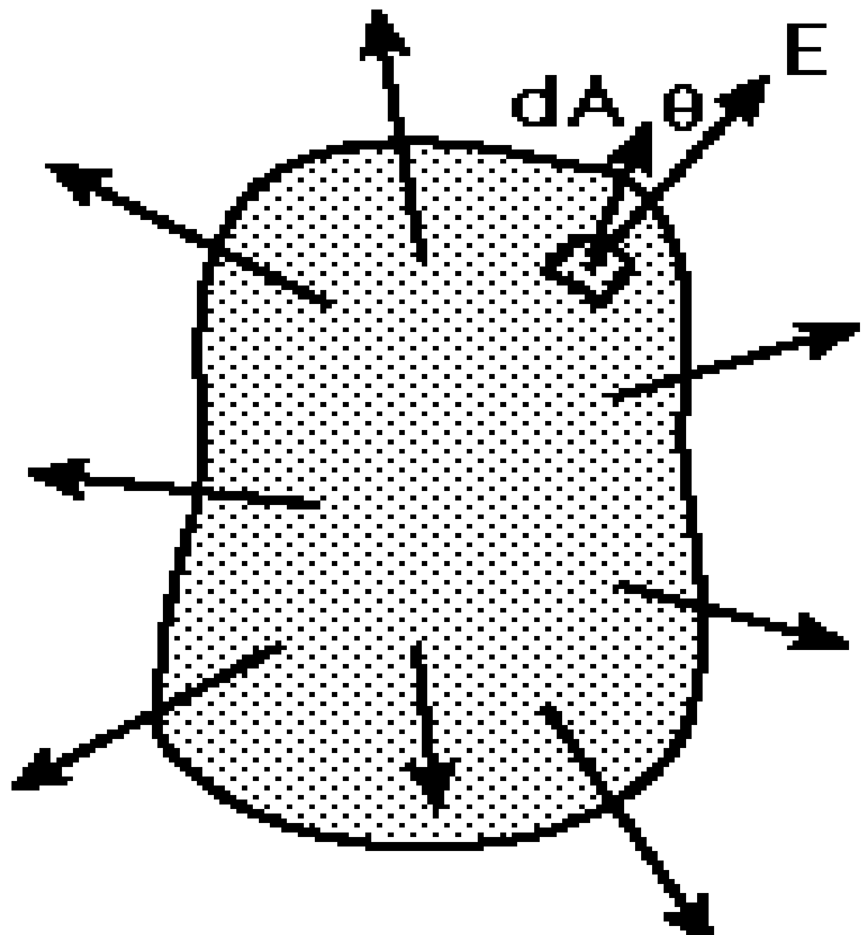


$$\Phi = EA\cos\theta = E_{\perp}A$$

If $E \perp$ surface, $\Phi = EA$

Flux Through A Closed Surface

Add up the flux through each small area



Sign Convention

Outward Flux: +

Inward Flux: –

Gauss's Law

$$\Phi = \frac{Q_{enclosed}}{\varepsilon_0}$$

Big Idea: Electric flux through a closed surface depends only on the charge enclosed by the surface

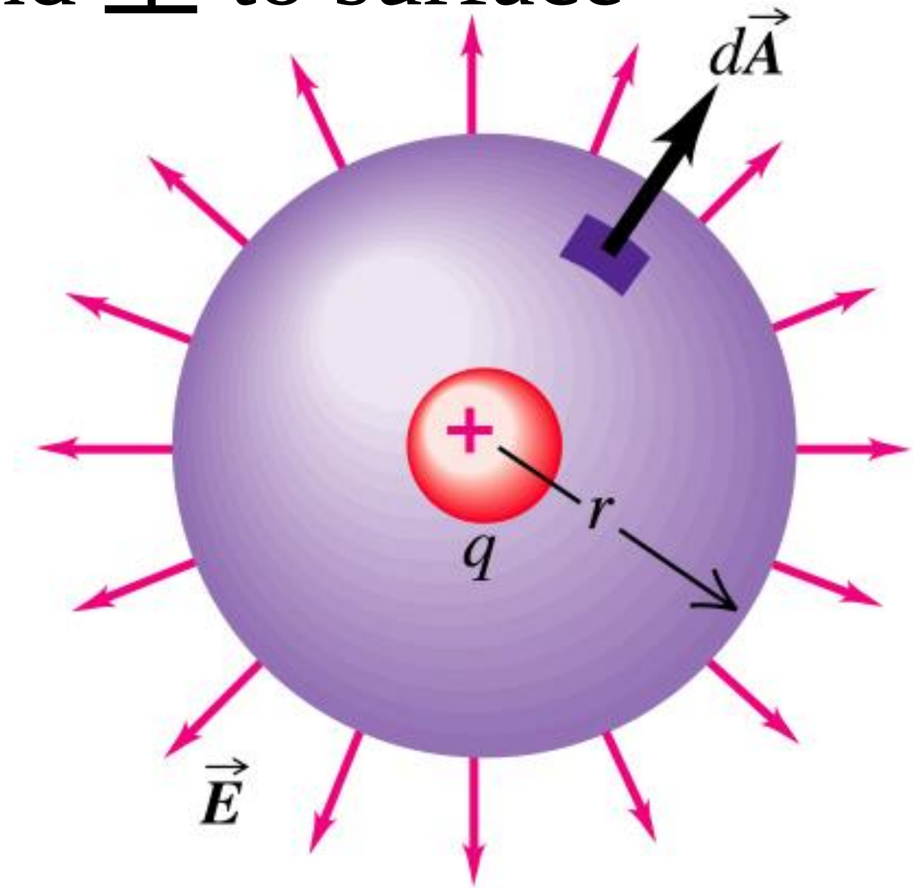
Example: Point charge at center of a sphere

KEY: **E** is uniform and $\perp$ to surface

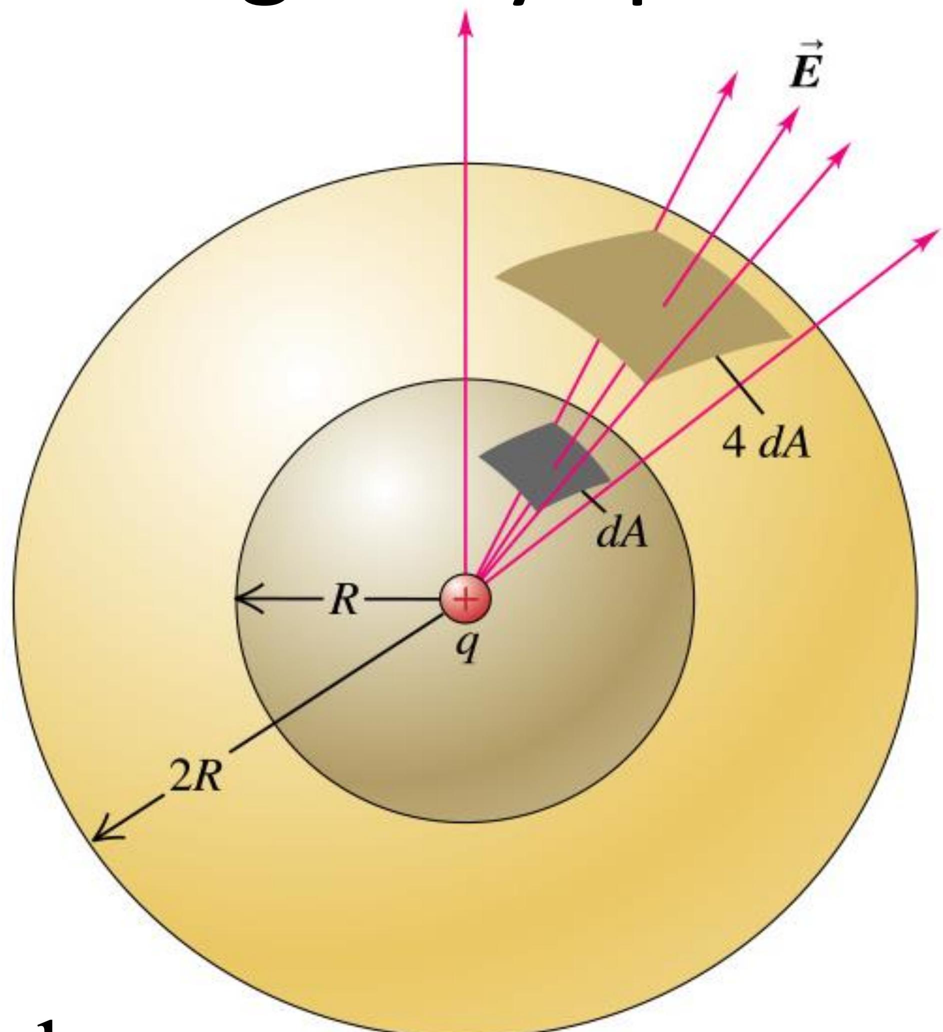
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$A = 4\pi r^2$$

$$\Phi = EA = q/\varepsilon_0$$



Flux is the same through any sphere!



Area $\propto r^2$

$E \propto \frac{1}{r^2}$

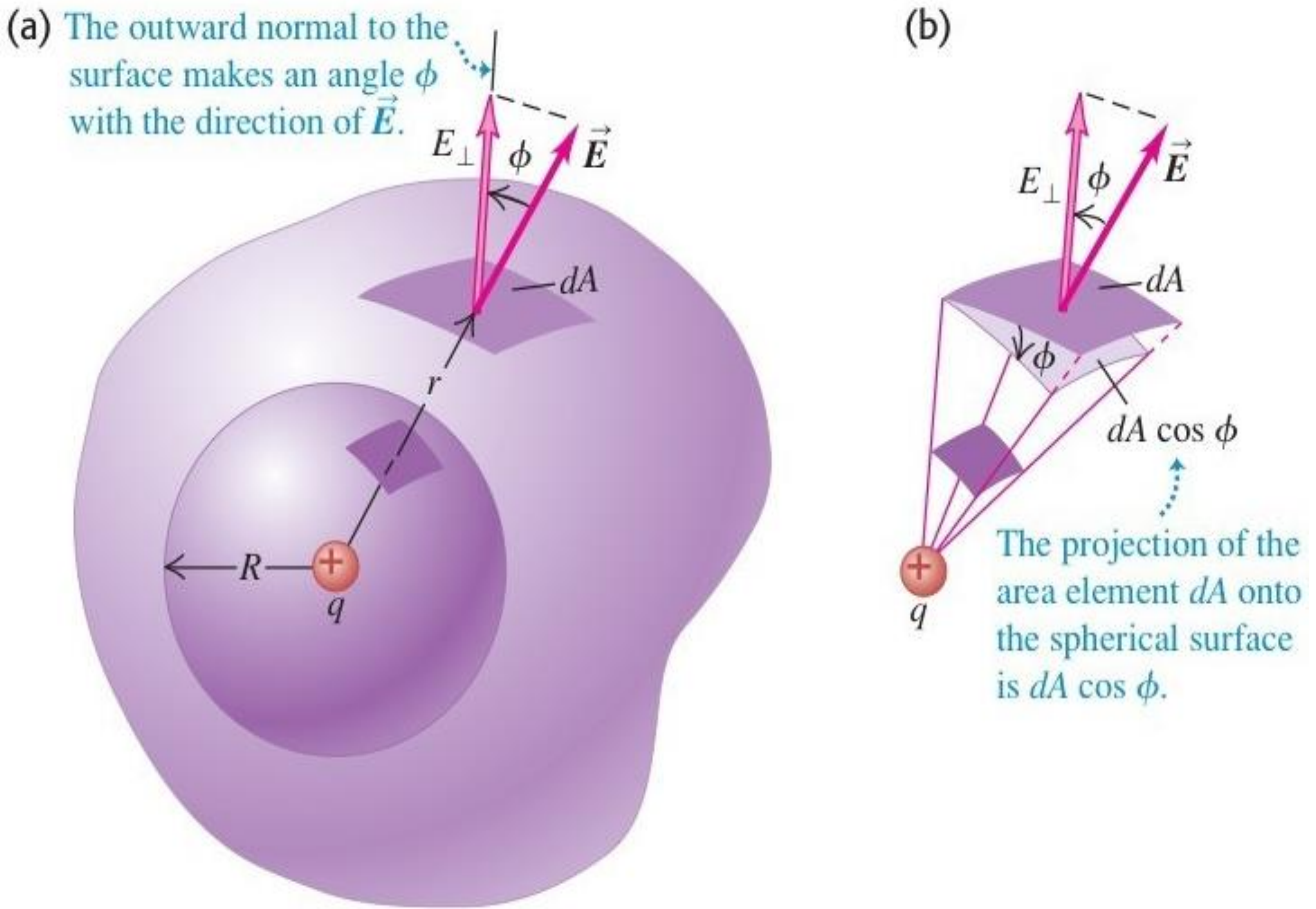
$\Phi = EA \rightarrow$ no r dependence

Flux through other surfaces?

Big Idea: Any surface can be created by deforming a sphere by

1) stretching/compressing

1) tilting


(a)
The outward normal to the surface makes an angle ϕ with the direction of $\vec{E}$.
$E_{\perp}$
ϕ
$\vec{E}$
dA
r
R
q
(b)
$E_{\perp}$
ϕ
$\vec{E}$
dA
ϕ
$dA \cos\phi$
q
The projection of the area element dA onto the spherical surface is $dA \cos\phi$.

Stretching has no effect on flux!

$dA \propto r^2$ and $E \propto \frac{1}{r^2}$

$$d\Phi = E dA \rightarrow \text{no } r \text{ dependence}$$

Tilting has no effect on flux!

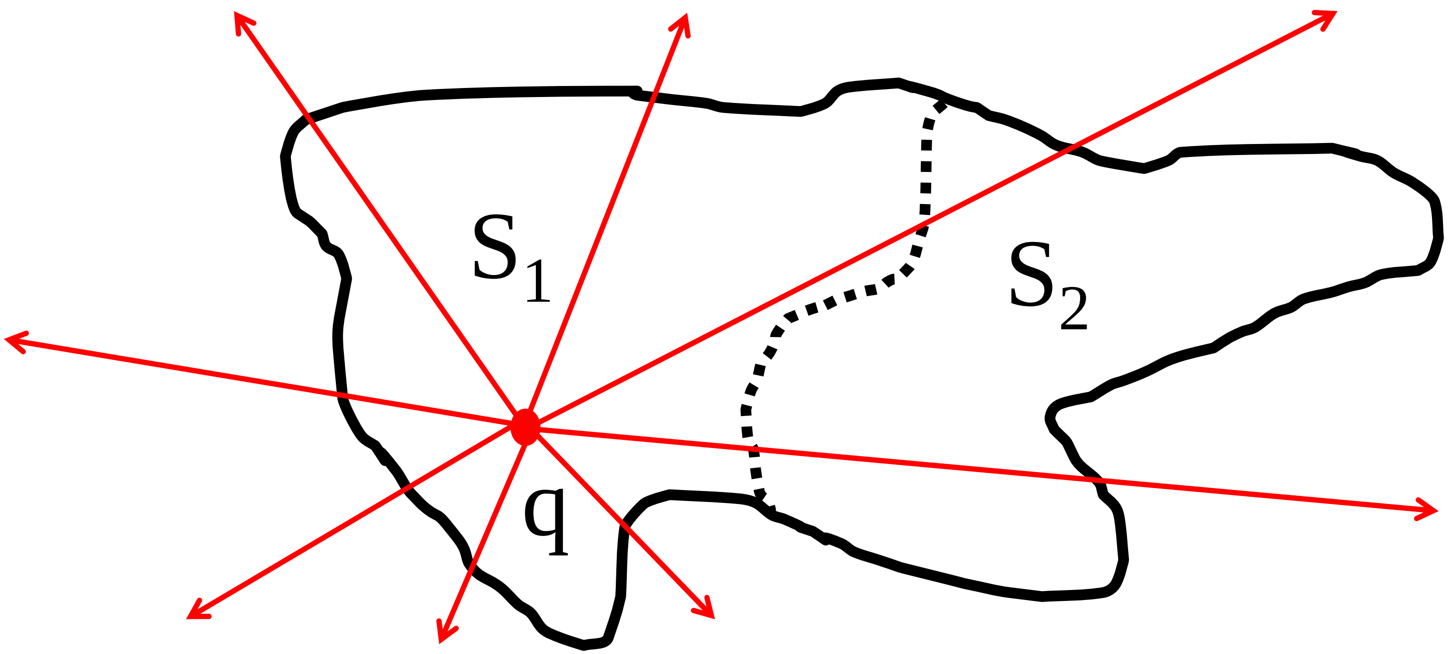
$dA \propto \frac{1}{\cos\theta}$ and $E_{\perp} = E\cos\theta$

$$d\Phi = E_{\perp} dA \rightarrow \text{no } \theta \text{ dependence}$$

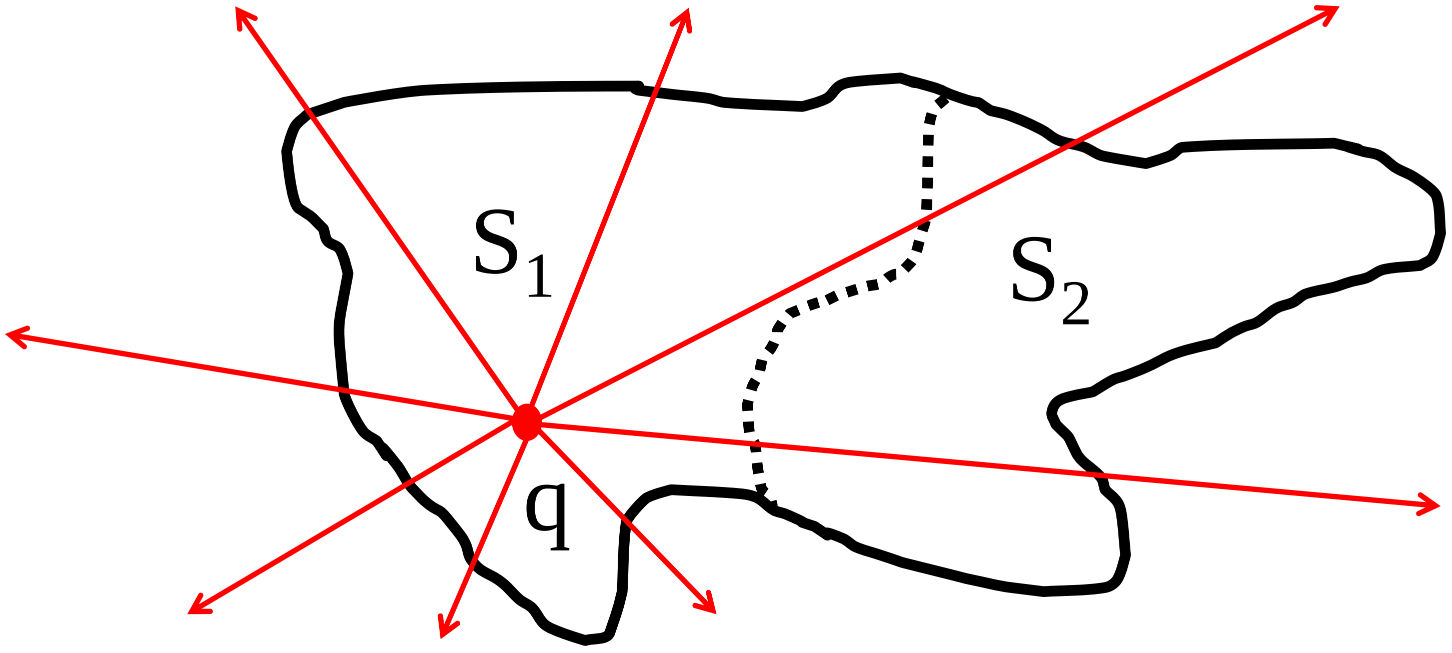
Conclusion: Electric flux through <u>any</u> surface enclosing an isolated point charge is the same!

What is the flux through a closed surface not enclosing charge?

Example: S_1 encloses q but S_2 does not



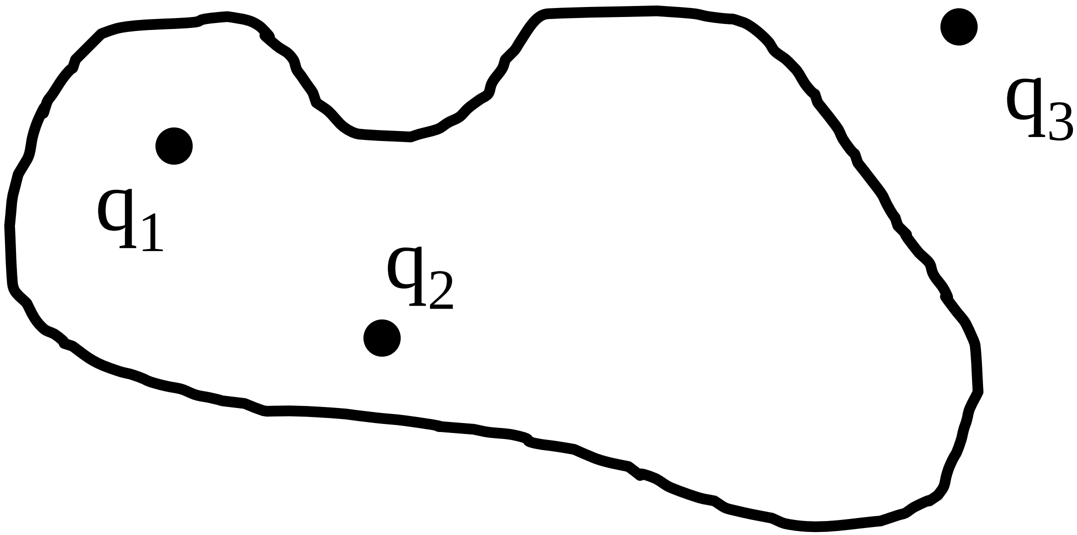
Let S be the entire surface: $S_1 + S_2$



$\Phi_S = \Phi_{S_1} + \Phi_{S_2}$ <u>but</u> $\Phi_S = \Phi_{S_1} = q/\varepsilon_0$

$\therefore \Phi_{S_2} = 0$ (flux into (-) & out of (+))

What if there are many charges?



$$\Phi = \oint \boldsymbol{E} \bullet d\boldsymbol{a} = \oint (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3) \bullet d\boldsymbol{a}$$

$$= \oint \boldsymbol{E}_1 \bullet d\boldsymbol{a} + \oint \boldsymbol{E}_2 \bullet d\boldsymbol{a} + \oint \boldsymbol{E}_3 \bullet d\boldsymbol{a}$$

$$= \quad q_1/\varepsilon_0 \quad + \quad q_2/\varepsilon_0 \quad + \quad 0$$

$$= \quad q_{enclosed}/\varepsilon_0$$

KEY: Gauss’s Law always holds but is most useful where there is lots of symmetry!

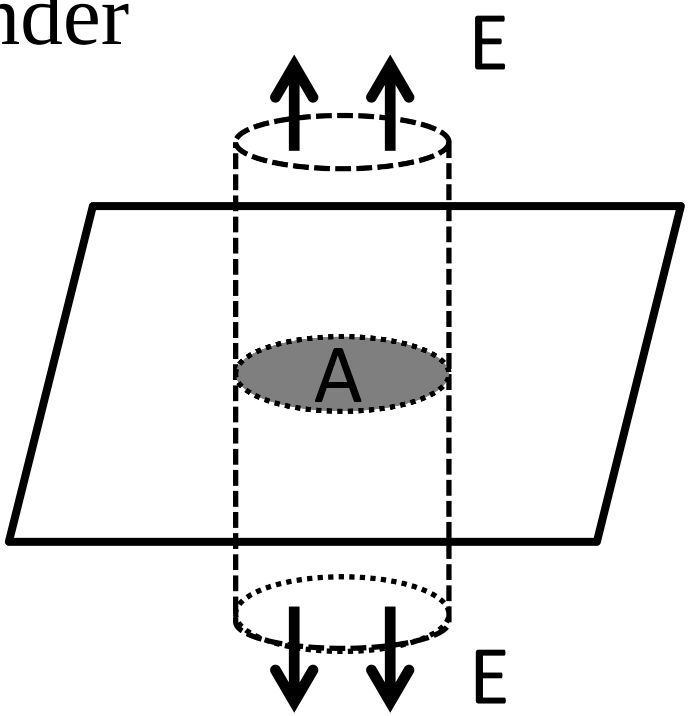
(i.e., where flux can be found without actually doing an integral)

Example: Find E for an infinite sheet of charge density σ (per area)

Gaussian Surface: Pill Box/Cylinder

-Charge enclosed (shaded):

$$Q_{enclosed} = \sigma A$$



-Flux (through two ends):

$$\Phi = 2EA$$

$$\Phi = Q_{enclosed}/\varepsilon_0 \rightarrow 2EA = \sigma A/\varepsilon_0$$

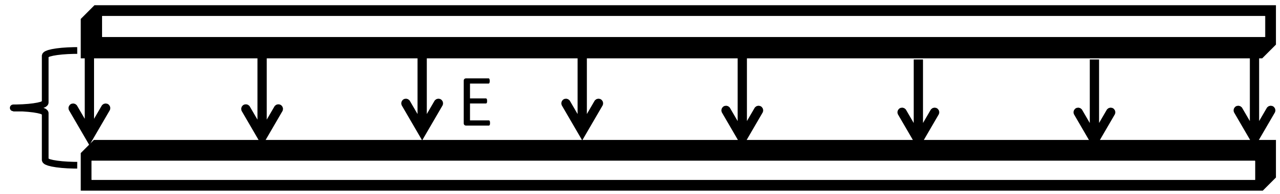
Result: $E = \sigma/2\varepsilon_0$

Ideal Parallel Plate Capacitor

Area of plate face: A

Charge on inner face: +Q

Plate separation: d



voltage: $V \approx Ed$

Charge on inner face: -Q

Treat faces like sheets of density: $\sigma = Q/A$

$$E_{total} \approx 2\ E_{sheet} = 2(\sigma/2\varepsilon_0) = Q/A\varepsilon_0$$

$$C = Q/V = Q/Ed = Q/(dQ/A\varepsilon_0) = \varepsilon_0 A/d$$

Summary: Parallel-Plate Capacitors

Ideal Formula: $C_0 = \varepsilon_0 A/d$

Derived from Gauss's Law by treating plates like infinite, uniform charge sheets (decent approximation for $d \ll \sqrt{A}$)

Actual capacitance differs!

- edge effects
- tilt effects
- body capacitance