General Physics - E&M (PHY 1308) Lecture Notes

Lecture 004: Continous Charge Distributions and Matter in Electric Fields

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no tags

Goals

- Discuss the transition from finite distributions of charge to continuous distributions of charge
- Discuss what happens to matter in the presence of electric fields

Continuous Charge Distributions

We have been dealing with systems with one or two charges, or treating more complex systems (e.g. the heart, on the problem set) like a single point charge. Today, we get tough. Today, we discuss the transitions from few charges to lots of charges.

Nature is build from elementary charges like electrons and protons. However, a typical number in a volume of air or in a solid is Avagadro's Number: 6.02×10^{23}! Good luck adding all of those forces and fields together. We need tools to attack these more realistic situations. We need calculus.

To begin, let's discuss the bulk properties of continuous charge distributions:

- *Volumes of charge*: given a volume (3 dimensions) of charge, we will speak of the **volume charge density**, ρ, whose units are charge per unit volume, C/m^3.
- *Surfaces of charge*: if the charge is spread out over a surface, instead of throughout a volume, we will speak of the **surface charge density**, σ, whose units are C/m^2.

- *Line of charge*: if the charge can be described as distributed along a single dimension, we will then speak of **linear charge density**, λ, whose units are C/m.

Using these concepts lets us summarize the charge distribution properties of an extended, continuous system. Instead of saying a charge is here, at some coordinates, and over there, at some other coordinates, we speak instead of densities of charge along a line, surface, or throughout a volume.

Cement the concept: examples

Imagine that I have a cube, whose sides are each of length $1.0\mathrm{m}$. In each of the following situations, tell me the appropriate bulk property of the system:

- What is the bulk property of the system if there is a charged object inside the cube whose strength is $10\mathrm{C}$? ANSWER: we should be concerned about the charge per unit volume, or the volume charge density of the system. In that case, it will be $\rho = (10\mathrm{C})/(1\mathrm{m})^3 = 10\mathrm{C}/\mathrm{m}^3$.
- What is the bulk property of the system if the same $10\mathrm{C}$ charge is distributed over the sides of the cube? ANSWER: we should be concerned about the charge per unit area, or the surface charge density. In that case, we need the surface area of the cube. That's $A = 6 \times (1\mathrm{m}^2) = 6\mathrm{m}^2$. Thus $\sigma = 10\mathrm{C}/6\mathrm{m}^2 = (5/3)\mathrm{C}/\mathrm{m}^2$.
- What is the bulk property of the system if the same $10\mathrm{C}$ charge is distributed only along the thin corners at each side of the cube? ANSWER: if the charge is isolated to just the corners where each side meets, you can imagine the cube as being made from a wireframe where each wire holds part of the total charge. Thus, the bulk property we are concerned with is the linear charge density. We need to know the lengths of these "wires". There are 12 such lines that make up the cube. Thus the total length is $L = 12 \times 1\mathrm{m} = 12\mathrm{m}$, and $\lambda = 10\mathrm{C}/12\mathrm{m} = (5/6)\mathrm{C}/\mathrm{m}$.

Summing the fields

We've used the Principle of Superposition to add together individual fields from a series of charges. Now that we're dealing with lots of charge spread out over lines, surfaces, and volumes, we need a better way to do the sums. Calculus - and specifically the integral - gives us this power.

Recall that an integral is just given in the limit of a sum of very small pieces of a problem. For instance, if we have a bunch of small charges and we want to find the total field at some point, P. Each infinitesimal unit of charge, dq, is responsible for emitting its own infinitesimal piece of the total field, $d\vec{E}$. Thus to obtain the total field we must integrate over the individual infinitesimal fields:

$$\vec{E} = \int d\vec{E}$$

We know how to write the electric field of each infinitesimal point of charge - that's just using Coulomb's Law! We have to do a transformation of variable in the integral, from $d\vec{E}$ to dq. That's simply done as follows:

$$d\vec{E} = \left(\frac{d\vec{E}}{dq}\right) dq = \left(\frac{d}{dq}\frac{kq}{r^2}\hat{r}\right) dq = \left(\frac{k}{r^2}\hat{r}\right)\frac{dq}{dq}dq = \frac{k\,dq}{r^2}\hat{r}$$

Thus:

$$\vec{E} = \int \frac{k\,dq}{r^2}\hat{r}$$

This is the easy part. The hard part is being given a physical problem and being asked to solve for the field in terms of the geometry of that problem. Attacking real problems will involve translating the geometry of the problem into something over which you can then integrate more easily. Let's look at an example.

Example: a line of charge

Consider a thin wire that carries a charge. The bulk property of the wire that we are interested in is the linear charge density, λ. Imagine that this wire - perhaps a power line or an electric cable of some other sort - lies along the x-axis. How do we find the electric field at some point P away from the line?

INTERPRET: we have to start somewhere. Let's treat the wire as so long that we don't have to worry about being close to its ends, where the geometry changes (and is not very line-like anymore). Thus, let's call this line infinite in length. Let's define our point P, then, as lying at a point in space $(x, y) = (0, y)$. Thus, it lies a distance y from the line along the y-axis. The source charge in the problem is the whole wire.

DEVELOP: Make a drawing of the problem. We divide the wire into small charge elements dq. NOTE: one thing we see right away is that if we consider two charge elements lying an equal distance x along the x-axis from $x = 0$, the electric fields of these two elements is equal and opposite alone the x-axis and cancels. This leaves only the y-components of the individual fieldsm $d\vec{E}$, to be considered. This immediately reduces the problem into one where we only have to find the y-component of each infinitesimal field! Thus we only need the y-component of each unit vector, $\hat{r}_y = y/r$.

Let's explore that last statement. The distance from any point on the wire to the point P is $r = \sqrt{x^2 + y^2}$, and the vector from that point on the wire to P is given by $\vec{r} = (x, y)$, where x is the location along x of the charge element dq, and y is the distance from the wire to P along the y-axis. How does one obtain the unit vector from this? Recall that generally speaking, $\hat{r} = \vec{r}/r$. Thus in our case, $\hat{r} = (x/r, y/r)$. Since we are only concerned with the electric fields along y, we only need the y-component of this unit vector: $\hat{r}_y = y/r$.

EVALUATE: We have pieces. Let's assemble them. We need to relate dq to some geometric variable so that we can perform the integration. We know that our wire has a linear charge density of λ. If a charge element has a length dx, then $\lambda = dq/dx$ or $dq = \lambda dx$. Thus: $dE_y = \frac{k\,dq}{r^2}\hat{r}_y = \frac{k\lambda\,dx}{r^2}\frac{y}{r} = \frac{k\lambda y}{(x^2+y^2)^{3/2}}dx$

Since the x-components all cancel out due to the symmetry of the problem, we can write the field as $\vec{E} = (0, E_y)$ where $E_y = \int dE_y = \int_{-infty}^{+infty} \frac{k\lambda y}{(x^2+y^2)^{3/2}}dx$. Have a look in Appendix A to see how to do this integral.

The result is: $E_y = \frac{2k\lambda}{y}$.

ASSESS: this is interesting. Since all the fields along x cancel, the electric field along a charged thin infinitely long wire simply radiates outward from the wire perpendicular to its length, with a strength that falls inversely with the distance y of the point P from the wire.

An infinite line is impossible, but this result does hold very well if you study long wires carrying an electric charge linear density and make measurements of the field far from either end of the wire. The results hold approximately as long as we're not much further from the wire than its length.

Matter in Electric Fields

We are constantly exposed to electric fields, whether they are the weak ones in phones and computers, the strong ones from lightning or carpet electric shocks, the useful ones that ignite natural gas for cooking or light our homes. We are matter, just as much of what is around us is matter. How does matter, as a whole, respond to electric fields?

Single charges

Let's start with a single charge, and work our way up. We can ask what a single electric charge will do when it is suddenly exposed to an electric field.

- *PhET* Simulation of Single Charge in Electric Field: http://phet.colorado.edu/en/simulation/efield

What will happen to a single charge that is exposed to an electric field?

We have some notion that an electric field and a force are related to one another. Applying Newton's Law that $\vec{F} = m\vec{a}$, we know that in the presence of a force a mass m will accelerate. Combining this with the definition of the electric field, $\vec{E} = \vec{F}/q$ for a test charge q, we learn that $\vec{a} = (q/m)\vec{E}$. That is, in the presence of an electric field from some source, a charge q with mas m will experience an acceleration given by this equation. Use the *PhET* simulator to test the prediction from this exercise.

This is also kind of cool, because it tells us that the amount of acceleration varies inversely with the mass. Low-mass particles - like an electron - experience a much HIGHER acceleration than high-mass particles (like a proton) exposed to the same electric field and carrying the same charge. In fact, this principle can be used to build a device that separates charged particles based on their mass: an electrostatic analyzer.

Applications: material analysis and space weather measurements

Less-simple example: dipole in an electric field

We discussed this qualitatively last time (see *PhET* demonstrator). A dipole in an electric field experiences equal but opposite forces on each end (due to the opposite charges at each end). As a result, the dipole rotates until it is aligned with the electric field. There is no NET force on the dipole, but it rotates (think about pushing with equal but opposite forces on opposite sides of a bike wheel. The wheel ROTATES, but it won't TRANSLATE (move along a straight line).

Rotational motion is something we learned to describe in mechanics, and we'll apply that here. A ROTATIONAL FORCE is a "torque",

$$\vec{\tau} = \vec{r} \times \vec{F}$$

It's a cross-product, where $\vec{r}$ is the distance from the center of the rotation to point of application of the force, and $\vec{F}$ is the force. Remember the right-hand rule! If you take your right hand, flatten it, and point your fingers in the direction of $\vec{r}$. Now curl your fingers into your palm in the direction of $\vec{F}$. Your thumb now indicates the direction of the torque, and is perpendicular to BOTH the $\vec{r}$ and $\vec{F}$.

The magnitude of a cross-product is given by:

$$\tau = rF\sin\theta$$

where θ is the angle between the vectors $\vec{r}$ and $\vec{F}$.

The torque tends to align the dipole with the field, in our case. Recall that we defined the *dipole moment* in the last class:

$$\vec{p} = (qd)\hat{d}$$

where q is the magnitude of the charge on either end of the dipole, d is the distance between the dipole charges, and $\hat{d}$ is a unit vector pointing from the negative to the positive charge. The magnitude is then:

$$p = qd$$

Since the vector $\vec{r}$ in this case has a length that is $(1/2)d$, we can rewrite the mechanical torque on the electric charge due to an electric field as:

$$\tau_+ = rF\sin\theta = (1/2)dF\sin\theta = (1/2)(p/q)F\sin\theta = (1/2)pE\sin\theta$$

The torque due to the force on the negative charge is:

$$\tau_- = (1/2)(p/-q)F\sin(\pi-\theta) = -(1/2)pE(-\sin\theta) = (1/2)pE\sin\theta$$

So the total magnitude of the torque on a dipole in a uniform electric field is:

$$\tau = pE\sin\theta$$

and the total torque is given by:

$$\vec{\tau} = \vec{p} \times \vec{E}$$

In a **uniform electric field**, the dipole experiences a torque but no net force. Because it takes work to rotate the dipole and align it with the field, energy is stored in the dipole. If the dipole begins at a right-angle to the field $(\pi/2)$, the work required to rotate it to a new angle θ is:

$$W = \int_{\pi/2}^{\theta} \tau d\theta = \int_{\pi/2}^{\theta} pE \sin\theta \, d\theta = pE[-\cos\theta]|_{\pi/2}^{\theta} = -pE\cos\theta$$

This work ends up as stored potential energy, $U = W$. Remebering that the **dot product** of two vectors, $\vec{a}$ and $\vec{b}$, has a magnitude equal to $ab\cos\theta$, where θ is the angle between the two vectors, we can write:

$$U = -\vec{p} \cdot \vec{E}$$

You see how useful is this notion of a dipole moment as a characteristic of the system.

Dipoles in non-uniform electric fields

When a dipole is in a non uniform electric field, like the water molecules in the field of a hair comb, a net force DOES result since each end of the dipole experiences a slightly different force. We saw that in the comb/water movie. This effect is extremely important in nature, especially because two dipoles are often next to one another (e.g. two water molecules). This weak net attractive force that happens between neighboring dipoles is called a Van Der Waals Force. You probably studied it in chemistry, but this is why is happens at all. Here's why it's important to understand the Van Der Waals force.

EXAMPLE: Your First Breath Could Have Been Your Last

Did you know that, as aerobic creatures with lungs, we are almost nearly killed by dipoles when we are born? Our lungs are extremely complex. Just before we are born, one of the last things that happens during development is the secretion of a fluid in the lungs - a "surfactant" - that coats the inner lining of the alveoli (c.f. "Pulmonary surfactant" in http://en.wikipedia.org/wiki/Pulmonary_surfactant). Without this surfactant, the thin lining of water in our lungs (remember, water is a dipole) would exert so much attractive dipole force on itself that after first exhaling, we would be UNABLE to inhale again. The surfactant reduces the attraction between water dipoles by bonding at one end with water molecules. This relieves the surface tension in our alveoli and allows us to breathe.

Premature babies, born before the surfactant can be naturally produced in sufficient quantities, have the surfactant introduced artificially into their lungs by an aerosol spritzed into their mouths. This allows them to breath normally. Introduction of the surfactant once is sufficient, since it's recycled at the 90% level in infants. My own twin nephews had this happen to them when they were born about 6 weeks premature.

Conductors, insulators, and dielectrics

We're going to talk a lot about different kinds of materials, and how they permit electric fields to pass through them.

- Materials where charges are free to move around are **conductors**
  - the motion of electric charge is called **electric current**, akin to water currents (the flow of water through itself or another medium)
- Materials where charges are NOT free to move are **insulators**
- A material that contains dipoles, either natively or when exposed to an electric field, serve to REDUCE the electric field inside the material. These materials are called **dielectrics** They permit electric fields, but in doing so weaken them. This is an important effect we'll explore more later.
  - What happens when a dielectric experiences too much field? Individual charges in the dipoles can be ripped free, and then become free to move in the presence of this extremely high electric field. This is **dielectric breakdown** and signals the failure of the dielectric material. Lightning, for instance, results from the dielectric breakdown of the air between the ground and the sky (where charge is building up on either end).