General Physics - E&M (PHY 1308) Lecture Notes

Lecture 006: Gauss's Law

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no tags

Goals of Today's Lecture

- Introduce Gauss's Law as a powerful means to infer the properties of an unknown charge distribution from the properties of its electric field

Gauss's Law

We've arrived at the moment when we can derive and then discuss one of the foundational, fundamental equations in physics. This equation is part of a set of a few equations - laws of nature - which formed the basis of the revolutions that led to quantum physics and relativity. But to get there, we have to derive, use, and try to understand this equation. It is called "Gauss's Law".

Remembering our definition of flux from last time,

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{A},$$

we recall that for a closed surface surrounding a distribution of charge the flux has something to do with the amount of enclosed charge. The net amount of flux emerging from the surface tells us about the magnitude of the charge enclosed. We also recognized from our visualization exercise that a surface that encloses no net charge also has no net flux.

Gauss's Law is the math that expresses this recognition, and its validity has been put to the test countless times. To derive Gauss's Law, let's imagine a point charge $q_{enclosed}$ enclosed by a simple surface - a 3-D sphere. Let's design the sphere such that the charge lies exactly at its center. The surface is characterized by the normals to it, and for a closed surface the normals all point outward. Therefore $d\vec{A} = \hat{r} dA$. We also know from

Coulomb's Law that for a point charge, the electric field ALSO points in the directions $\hat{r}$. Therefore $\vec{E}$ and $d\vec{A}$ are parallel, $\cos\theta_{EA} = 1$, and we have simple $\vec{E} \cdot d\vec{A} = E\,dA\,\cos\theta_{E\,dA} = E\,dA$. We can then write the flux as:

$$\Phi = \int_{surface} E\,dA = E(4\pi r^2)$$

where the integration of little pieces of spherical surface, done over the whole surface, simply yields the surface area of the sphere: $A = 4\pi r^2$.

Thus we arrive at:

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{A} = \frac{k\,q_{enclosed}}{r^2}(4\pi r^2) = 4\pi k q_{\mathrm{enclosed}}$$

.

For the sake of simpler notation, let us define a useful constant: $\epsilon_0 \equiv 1/4\pi k$ and state that we'll use this form of k from now on. It has units $\epsilon_0 = 8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$.

Then we have arrived at Gauss's Law:

$$\int_{surface} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0}$$

Generalizations from Gauss's Law

We see the math expresses something we inferred from our visualization efforts on Wednesday:

- To describe the charge enclosed in a surface, we need only understand how the field behaves across that surface - the flux
- If the surface encloses no charge, then the flux will net to zero.
- Gauss's Law and Coulomb's Law are really the same thing - they encode the same information.
    - For instance, if you increase the size of your spherical surface by a factor of 2, then the area $A = 4\pi r^2$ increases by a factor of 4. However, the electric field strength decreases - also by a factor of 4. In the end, the flux is the same *regardless of the size of the surface* If the inverse square law didn't hold, Gauss's Law wouldn't

hold either.

Even though we've been dealing with a simple case - a point charge - Gauss's Law is the same regardless of the charge distribution that is enclosed.

Problem Solving: Gauss's Law

If a point charge is enclosed by a spherical Gaussian surface, and another point charge is placed nearby but OUTSIDE the Gaussian surface, what is true of the total flux through the sphere. Does it increase, decrease, or remain the same? ANSWER: it remains the same, because the charge enclosed is unchanged.

What happens to the strength of the field at a point on the sphere directly between the two charges? Does it go up, down, or stay the same? ANSWER: it goes up. But, correspondingly, the field strength DECREASES on the opposite side of the sphere. In the end, the integral over $\vec{E} \cdot d\vec{A}$ remains unchanged.

To tackle a problem using Gauss's Law, do the following:

- INTERPRET: Is Gauss's Law the best way to tackle the problem? To answer this, think about whether the problem has sufficient symmetry to make Gauss's Law the sufficient choice to find the electric field. What is the symmetry in the problem? Is it spherical, a line, or a plane? If such a symmetry does not exist, while Gauss's Law always holds it may not be possible to use it.

- DEVELOP: make a diagram of the charge distribution. Use the symmetry to infer the direction of the electric field. Then draw an appropriate *gaussian surface* - an imaginary surface that encloses the distribution and will let you evaluate the integral in Gauss's Law. The field should have constant magnitude on the surface and should be perpendicular to the surface. Sketch some field lines - the symmetry should indicate their direction. If you can't find a suitable Gaussian surface, there may not be sufficient symmetry to apply Gauss's Law.

- EVALUATE:
  - Evaluate the integral, $\int_{surface} \vec{E} \cdot d\vec{A}$ on the gaussian surface. Since the surface is designed to get $\vec{E}$ parallel to $d\vec{A}$, $\cos\theta = 1$. With the field having constant strength at the Gaussian surface, you can pull E out of the integral. If $\vec{E}$ is parallel to the surface, as happens in line or plane symmetry, then there are no contributions from those locations.
  - Evaluate the *enclosed charge*. This may or may not be the same as the total charge.
  - Evaluate the electric field by invoking Gauss's Law. The direction of the field should be evident from symmetry.

- ASSESS: Does your answer make sense? Think before you box the answer. Does the field behave as you except, given what you know of simpler charge distributions - points lines, etc?