General Physics - E&M (PHY 1308) Lecture Notes

Lecture 007: Applications of Gauss's Law and Conductors

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no tags

Goals

- Learn to work problems involving Gauss's Law

Working with Gauss's Law

As in all problems in physics, all the work is in setting up the question. If you can narrow down on what's being asked, and how to translate that into the language of mathematics (sprinkling in some physics thinking for good measure), then you will be able to setup and solve hard problems. I'll work two problems today to illustrate this.

Applying Gauss's Law: A Uniformly Charged Sphere

Imagine that you construct a spherical structure of charge, where the charge is uniformly distributed throughout the volume, whose radius is R. Can you find the electric field at all points?

To attack this problem, we can first consider the geometry. We have some charge Q spread uniformly throughout this sphere. That means we have a *constant volume charge density*, ρ. Independent of the sub-volume, dV, of the sphere we consider, $\rho = dq/dV$ will be the same. One can already suspect that this will come in handy for rewriting charge in termed of something we can actually integrate.

There are really two "regions of interest" in this problem: points inside the sphere, and points outside the sphere. The distribution has spherical symmetry, and so we should be able to "easily" use Gauss's Law to solve for the field.

Gauss's Law works best when you have symmetry of some kind: spherical, planar, linear.

Let's begin by writing down Gauss's Law:

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{A} = q_{enclosed}/\epsilon_0$$

Let's then consider two Gaussian Surfaces: a spherical surface inside the charged sphere, with its own radius r ($r < R$), and a spherical surface that encloses the entire charge sphere and with $r > R$. Just given the symmetry of this sphere, you can already see that whatever the electric field, its vectors will point radially outward from the center of the charge-sphere.

We need to attack that flux integral on the left-hand side. We can use what we have just observed about the charge-sphere to start simplifying things:

- First, given the spherical symmetry of the problem, we already know that $\vec{E}$ is parallel to the normal pointing out from our Gaussian surface piece, $d\vec{A}$. Thus in the dot product, $\vec{E} \cdot d\vec{A} = EdA\cos\theta = EdA$ since $\cos\theta = 1$. So the flux integral just boils down to $\int_{surface} EdA$.
- With our Gaussian surface centered on the charge sphere, it's the same distance from all points along the same radius on the enclosed charge - therefore whatever the field, its value is *constant across the Gaussian surface* and thus can come out of the integral, $\Phi = E\int_{surface} dA$.
- The integral $\int_{surface} dA$ is going to simply yield the total surface area of a sphere of radius r, the radius of our Gaussian surface. Thus $\Phi = EA = E(4\pi r^2)$.

And we've evaluated the surface integral.

Now we need to evaluate the right-hand side of Gauss's Law: $q_{enclosed}/\epsilon_0$. We need to determine the amount of charge enclosed by our Gaussian surface. It will matter whether $r < R$ or $r > R$. For $r > R$, the answer is straight-forward: $q_{enclosed} = Q$, the whole charge of the sphere. What about for $r < R$?

- To tackle this, we need to take advantage of the constant volume density of charge; that is, ρ is the same regardless of the volume inside the sphere we choose to evaluate. Let's consider two different ways of obtaining ρ. We could consider the whole volume of the charge-sphere. In that case,

  $$\rho(R) = \rho = Q/((4/3)\pi R^3).$$

  We could also consider the Gaussian sphere for which $r < R$. In that case, [rho(r) = rho = q_{enclosed}/((4/3) pi r^3).] These two ρs are equivalent, so setting them equal we find that

  $$Q/((4/3)\pi R^3) = q_{enclosed}/(4/3)\pi r^3) \to q_{enclosed} = Q\left(\frac{r}{R}\right)^3.$$

Let's put all of these pieces together - the right-hand side of Gauss's Law and the left-hand side of Gauss's Law:

- For $r < R$:

  $$\int_{surface} \vec{E}\cdot d\vec{A} = q_{enclosed}/\epsilon_0 \to E(4\pi r^2) = Q(r/R)^3(1/\epsilon_0).$$

  This reduces to

  $$E(r < R) = \frac{Qr}{4\pi\epsilon_0 R^3}$$

- For $r > R$:

  $$E(4\pi r^2) = Q/\epsilon_0$$

  which reduces to

  $$E(r > R) = \frac{Q}{4\pi\epsilon_0 r^2}.$$

Draw the function on the board, with the linear piece for $r < R$ and the inverse-square law piece for $r > R$.

Does this make sense? Inside the sphere, as we grow the size of our Gaussian surface, we include more and more charge and we expect the field strength to grow. Outside the sphere, we fully enclose all the charge and just get further from the charge as r gets bigger, so we expect this to converge at large distances to the electric field of a point charge, Q, which it does.

Imagine how hard it would be to calculate this using just the superposition principle! It's a very beautiful and useful result. Incidentally, this law and this solution holds for gravity as well. It's why we can treat planets as "point masses" when were are outside of the planet's surface (as if all the mass were concentrated at the center of the planet in a point).

But what if our charge is NOT uniformly distributed in the interior?

Applying Gauss's Law: A Thin, Hollow Spherical Shell

Imagine a thin spherical shell of charge, with an outer radius R. It's thin enough that we can neglect its thickness for this problem. What is the electric field inside and outside the shell?

Again, this is a spherically symmetric distribution. Outside the shell, all of the charge is enclosed again an the field outside looks just like it did for the solid sphere of charge. We just have to concern ourselves with the field INSIDE.

- Construct a spherical Gaussian surface with radius $r < R$ centered on the shell's center. Again, for all the same reasons as in the charged sphere the flux is just $\Phi = \int_{surface} \vec{E} \cdot d\vec{A} = E(4\pi r^2)$.
- We're left to just evaluate the right-hand side of Gauss's Law. The gaussian surface encloses no charge, so $q_{enclosed}/\epsilon_0 = 0$.
- Therefore, $E = 0$ everywhere inside the shell!

Due to the inverse square law, at any point inside the shell the fields from charges on the side of the shell closer to the point are exactly cancelled by all of the fields from charges on the side further away. This is a remarkable fact, one which can be tested through experiment (we'll discuss this later).

But it also points to the answer to why our friend in the Boston Science Museum Lightning demonstration was not killed by the blasts from the Van de Graaff generator: charge deposited from the lightning bolt on the cage covered the surface of the cage due to the conductive nature of the metal (we'll discuss why this is shortly). The field inside the case, despite the massive bolt of lightning, was zero everywhere, and thus he was under no threat of physical harm.

One last problem, a revisit of an earlier one: a line of charge.

Applying Gauss's Law: A Uniform Line of Charge

This problem was kind of a pain when we did it with Coulomb's Law and the integral $\vec{E} = \int d\vec{E}$. Let's tackle it with Gauss's Law.

A line of charge has cylindrical symmetry, since on the surface of a cylinder centered on the line, it doesn't matter where you are on the cylinder - the electric field strength will be the same because you're equidistant from all points on the line.

- Symmetry requires that the electric field from a line of charge point radially outward, perpendicular to the line. Thus $\vec{E} = E\hat{r}$.

Let's make our Gaussian surface a cylinder.

- Define the Gaussian surface as a cylinder whose radius is r from the long axis and which has a length L. This allows us to attack the surface integral and get the flux. Consider the contributions to the flux from the ends of the cylinder: since the normals to the ends point either in the $+x$ or $-x$ direction, perpendicular to the electric field, there is *no flux contributions from the ends of the cylinder*.
- Consider the barrel of the cylinderical Gaussian surface. The normals to the surface all point along the cylinder radius, just like the electric field, so

  $$\Phi = \int_{surface} E \cdot d\vec{A} = E \int_{surface} dA$$

  and we just have to sum up the little pieces of cylinder area to get the total area of the cylinder. The area of a cylinder, which is just a rectangle of length L and height $2\pi r$ (where r is the radius of the cylinder) wrapped into a tube, is given by $A_{cylinder} = 2\pi r L$. Thus

  $$\Phi = E(2\pi r L).$$

- Now we just need to determine the right-hand side of Gauss's Law,

$q_{enclosed}/\epsilon_0$. We know that the line has a uniform charge distribution, and so $\lambda = q_{enclosed}/L$ considering the Gaussian surface and the charge enclosed by the cylinder. Thus

$$q_{enclosed}/\epsilon_0 = \lambda L/\epsilon_0.$$

- Finally, we set the left and right sides of Gauss's Law equal and find

  $$E(2\pi r L) = \lambda L/\epsilon_0$$

  which yields

  $$E = \frac{\lambda}{2\pi r \epsilon_0}.$$

  If we recall that $\epsilon_0 = 1/4\pi k$, then we recover the same solution as when we applied Coulomb's Law:

  $$\vec{E} = E\hat{r} = \frac{2k\lambda}{r}\hat{r}$$

Complex Charge Distributions and Gauss's Law

Gauss's Law works best when you can identify a symmetry in a problem and exploit that symmetry to solve the surface area integral (the flux integral). The book discusses what Gauss's Law tells us about planes of charge, dipoles, etc. Without clear symmetry, or without reduction of a problem to an illustrative simpler case or set of cases, it's hard to make Gauss's Law work for you. The book discusses the example of a disc of charge. Up close, a disc looks like a plane, and so you can say that very near the disc the field looks like that of a plane of charge. Far from the disc, it should resemble a point of charge electric field. There is some transition region in the middle that's harder to compute, but you know the transition between the plane field and the point field must occur.

Never underestimate the value of simple approximations. There is a joke about physicists that illustrates the point. An engineer thought it would be funny to ask her physicist friend to determine the gravitational field around a cow, knowing that the complex shape of the cow would immediately stump the physicist. The physicist thought about the question for a moment, turned to the board and started drawing, saying, "OK, let's consider a spherical cow of uniform density . . . "