General Physics - E&M (PHY 1308) Lecture Notes

Lecture 016: The Magnetic Field due to Moving Charge

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no tags

What is Magnetism?!

It was Hans Christian Oersted who is credited with setting physics and chemistry on the path to an understanding of magnetism. While magnetism has been observed since about 500-600 BC (in the Western world, it was Aristotle who gave credit for the discovery of this phenomenon to a man named Thales), it was not at all understood until the 1800s. Oersted accidentally observed that an electric current caused a compass needle to deflect. We can easily reproduce this experiment. Magnetism, therefore, seems to have something to do with the MOTION of electric charge. Not long after Oersted's publicized observation, two French scientists - Baptiste Biot and Felix Savart - performed experiments and determined the exact form of the force law for a steady current. We call this the *Biot-Savart* Law, and we'll explore it now.

The *Biot-Savart* law considers a steady current moving through a conductor. There are similarities and differences between *B-S* and Coulomb's Law. Let's write Coulomb's Law for a small piece of a distribution of charge:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$

Draw a picture representing the situation that can be described by Coulomb's Law (a blob of charge, considering the electric field due to a piece of the blob).

Now draw a picture representing the situation we want to analyze in magnetic fields. We want to know the field, $\vec{B}$, at a point P some distance, r, from a part of the conductor ($d\vec{L}$) carrying a steady current I. Our convention will again be that $\hat{r}$ points from the conductor element to the point P.

Now let's think about the differences between the static charge situation considered by Coulomb's Law and the moving charge situation in *B-S* Law:

- In the BSL, we are considering a current element, $Id\vec{L}$, which is a vector quantity (it flows along a direction in the conductor). In CL, we were considering a piece of charge which had no direction. It was just a number.
- In the BSL, the source of magnetic field is a VECTOR quantity - current times $d\vec{L}$. In the CL, the source of electric field is a scalar quantity - charge. We have to account for direction of motion in the BSL.
- In the BSL, the field contribution of $Id\vec{L}$ depends on the orientiation of the conductor to the unit vector - it depends on the sine of the angle, specifically. In CL, we had no such oddity.

The *B-S* Law, which describes all of our observations, is as follows:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{L}\times\hat{r}}{r^2}$$

Here, we have a new constant that has been determined from experimental measurement: μ_0, the **permeability constant**. It's EXACT value is $4\pi\times 10^7 \mathrm{N}/\mathrm{A}^2$. Equivalent units are often used: $\mathrm{T}\cdot\mathrm{m}/\mathrm{A}$.

There is one other important distinction between the BSL and CL. The CL gives us the electric field in terms of isolated charge elements. But it's impossible to talk about an isolated current element, because it necessarily must be part of a circuit. In order to get the total magnetic field, you have to integrate around the entire circuit to get the magnetic field at point P. Because magnetic field is a vector, it obeys the superposition principle so we just have to add up all the current elements:

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int\frac{Id\vec{L}\times\hat{r}}{r^2}$$

The above is the *integral form* of the BSL. The magnetic field thus depends on the details of the current distribution. Generally speaking, though, the cross-product in this law tells us that magnetic field lines encircle the path of the current perpendicular to its direction. Here we have another version of the right-hand rule:

- To determine the direction of magnetic fields around a conductor, point your thumb in the direction of current. The direction your fingers would curl around the conductor indicates the direction of magnetic field lines.

Example: magnetic field around a straight conducting wire

Consider an infinitely long straight wire carrying a steady current I. Find the magnetic field at a point P which lies a distance y above the wire.

- Draw the wired and setup the problem. Use the new right-hand rule to note which direction (into or out of the board) we expect the field to point.

Begin by writing the BSL:

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{Id\vec{L} \times \hat{r}}{r^2}$$

- Let's choose CARTESIAN coordinates since we have all these handy straight lines in the problem and because whatever the distance y of point P above the line, it's fixed no matter where along the conductor we are considering.

What are the unknowns?

- We need to sort out expressions for $\hat{r}$, $d\vec{L}$, and the cross-product in terms of the geometry (coordinates) of the problem
- We need an expression for r^2 in terms of geometry.

Let's attack pieces. What are they? They are: (1) the distance, r, (2) the relationship between dL and the geometry of the coordinate system, (3) the direction of the magnetic field due to $d\hat{L} \times \hat{r}$, and (4) the magnitude of $Id\vec{L} \times \hat{r}$.

- r or r^2 is the easiest: it's just $r^2 = x^2 + y^2$
- Before trying to compute the cross-product from its pieces, let's think about it a bit:
  - Can we figure out the unit vector for $\vec{B}$? Both $d\vec{L}$ and $\hat{r}$ lie in the plane of the board. Therefore, $\vec{B}$ must point into or out of the board. From the *R-H* rule, we expect it to point OUT - always perpendicular to both $\hat{r}$ and $\vec{L}$. There - we've figured out the direction of the cross-product without multiplying a single thing.
  - We can simplify the cross product magnitude as a result of the previous observation, since we know it points out of the board. Thus $|Id\vec{L} \times \hat{r}| = IdL \sin\theta$ where θ is the angle between $d\vec{L}$ and $\hat{r}$.
    - From the geometry of the problem, we know from trigonometry that $\sin\theta = \sin(\pi - \theta) = y/r = y/\sqrt{x^2 + y^2}$.
- What about dL? Well, the way we've setup the problem, that's just dx.

We have all the pieces. Let's re-write the BSL:

$$|\vec{B}| = \frac{\mu_0}{4\pi} \int \frac{I|d\vec{L} \times \hat{r}|}{r^2} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dx \sin\theta}{(x^2 + y^2)} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{y dx}{(x^2 + y^2)^{3/2}}$$

We can pull y out of the integral since we're not integrating it, and we're left with:

$$B = \frac{\mu_0 I y}{4\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + y^2)^{3/2}} \hat{k} = \frac{\mu_0 I}{2\pi y}$$

- What have we learned? We have learned that like the electric field from a line of static charge, the magnetic field from a line of moving charge falls of linearly with distance from the wire. But where the electric field lines point OUTWARD from the line, the magnetic field circles AROUND the line.

Magnetic attraction of two wires

What is you place two parallel lines of current next to one another, separated by a distance d? We know from the example above that the magnetic field of line 1 at the location of line 2, a distance d away, will be:

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

The field from line 1 will be perpendicular to the current in line 2, by construction, and we can just figure out the force immediately from:

$$F_{12} = I_2 L_2 B_1 = I_2 L_2 \frac{\mu_0 I_1}{2\pi d} = \frac{\mu_0 I_1 I_2 L_2}{2\pi d}$$

If the currents point in the same direction, the force between them is attractive. If they point in opposite directions, it's repulsive.

This force can be QUITE LARGE for large currents. Engineers have to worry about this force when designing electrical transport systems. The hum you often hear near high-voltage/high-current devices is due to the vibration from this magnetic force, as the current in such devices alternates (changes sign) and thus the force changes strength at about 60 times per second (60Hz).

Goals of this Lecture

- Introduce the electric current archetype: the loop
- Discuss the behavior of current loops in magnetic fields
- Motivate the atomic origin of macroscopic magnetism

The Magnetic Field from a Current Loop

An infinite line of charge is not very realistic, but a current loop is very realistic.Current loops are everywhere - remember circuits? Let's apply the *B-S* law to a simple circular loop of current. The results from this exercise will be our archetype for two things:

- the magnetic fields emitted by loops of current
- the geometry of loops immerse in external magnetic fields

Let's find the magnetic field due to a circular current loop along the axis of the circle. This is covered in Example 26.4 in Wolfson.

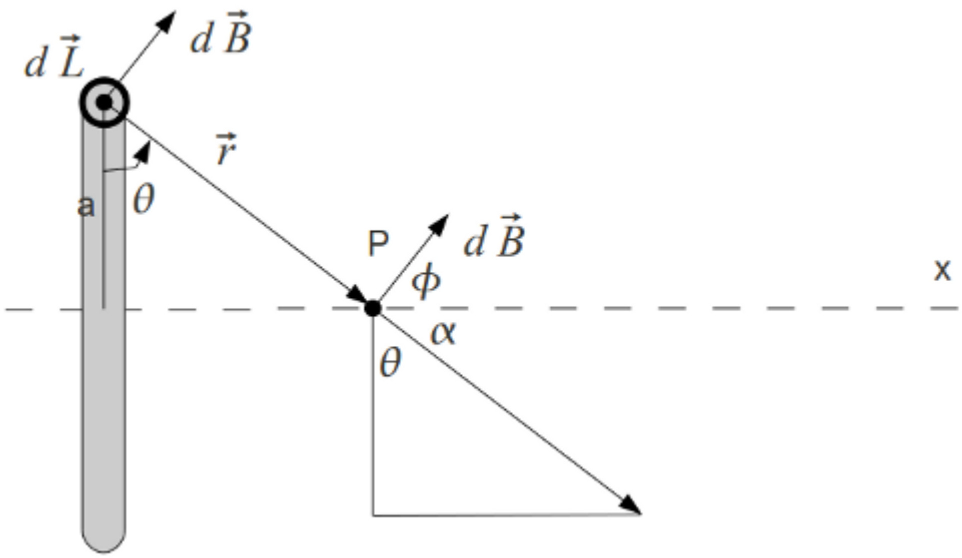
Let's begin by drawing the loop. Consider a point P, where we measure the field, that is a distance x from the plane of the circle. Current, I, is flowing counterclockwise through the loop. The radius of the loop is a.

- Begin by choosing a coordinate system. This problem is fully 3-dimensional, so we need either Cartesian or Spherical coordinates. Let's choose Cartesian, since we've not dealt with spherical coordinates yet in any serious detail. Let's define the x-axis as the axis along which we want to measure the field; the y-axis can then point up, splitting the circle in half, and z points outward.

- We need to express $d\vec{L} \times \hat{r}$, $d\vec{L}$, and r^2 in terms of the coordinates.
    - The only component of the magnetic field that survives along the x-axis is the x-component. This is due to symmetry. For every current element of the circle we choose, there is another element on the opposite side of the circle that has a $d\vec{B}$ whose component off the x-axis cancels that same component from the original element. Thus we only need to solve for $dB_x = dB\cos\phi$, where ϕ is angle between $d\vec{B}$ and the x-axis (see drawing).
        - How do we express ϕ? Is it related to anything else in this problem? I find this to be the most challenging part of the problem, actually. Since, when we get to optics, we'll need to dust off our trigonometry, let's attack this problem in detail. Consider the illustration of an x-y slice of the problem below. We see that the angle between the magnetic field piece, the current element, AND the distance vector is 90-degrees ($\pi/2$). If we then draw the magnetic field piece at point P, we have our unknown angle ϕ between it and the x-axis. Extending the right triangle symmetrically to the right of point P, we see that there is also an angle α between the mirror hypotenuse and the x-axis. From the properties of a right triangle, we know that $\alpha = \pi/2 - \theta$. We want to solve for ϕ. We see that $\alpha + \phi = \pi/2$. Thus, we find that $\phi = \theta$.

- We now know that this is the same θ as between the $\vec{r}$ and the radius of the circle. Thus, $\cos\theta = a/r = a/\sqrt{x^2+a^2}$.

- Let's consider the cross-product. By construction, the angle between the $\vec{r}$ and $d\vec{L}$ is 90-degrees. Thus we already know that $|d\vec{L} \times \hat{r}| = dL$.
- Finally, we need r^2. We already have that: $r^2 = a^2 + x^2$, from the right-triangle relationship.



We can now assemble all of these pieces into the BSL and solve for B_x:

$$B_x = \int dB_x = \int_0^L \frac{\mu_0}{4\pi} \frac{IdL}{x^2+a^2} \frac{a}{\sqrt{x^2+a^2}}$$

There is no dependence on dL anywhere in the function. Thus:

$$B_x = \frac{\mu_0}{4\pi} \frac{a}{(x^2+a^2)^{3/2}} I \int_0^L dL = \frac{\mu_0}{4\pi} \frac{a}{(x^2+a^2)^{3/2}} IL = \frac{\mu_0}{4\pi} \frac{2\pi I a^2}{(x^2+a^2)^{3/2}}$$

and we get our final equation:

$$B_x = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}\hat{i}$$

If we dig back in Wolfson to page 335, Example 20.5 - the field of an electric dipole along the bisecting axis of the dipole - we see something VERY interesting:

$$\vec{E} = -\frac{2kqa}{(x^2+a^2)^{3/2}}\hat{i}$$

In the special case that $x >> a$,

$$\vec{E} \approx -\frac{2kqa}{x^3}\hat{i}$$

and is proportional to $1/x^3$. In our case for the magnetic loop, when $x >> a$:

$$\vec{B} \approx \frac{\mu_0 I a^2}{2x^3}\hat{i}$$

and is ALSO proportional to $1/x^3$. Thus the magnetic field of a current loop - the simplest circuit we can make - behaves just like the electric field of a dipole. This is why we call this field a "dipole field", and this is also why we think that the simplest magnetic field is a dipole field.