General Physics - E&M (PHY 1308) Lecture Notes

Lecture 019: Ampere's Law

YourName, 4 April 2011 (created 4 April 2011)

no tags

Ampere's Law

What if the current being considered is not SIMPLE? What if it's a complex flow of electric charge over a vast span of area or volume?

Show the video of the solar flares illustrating the complex electric charge and magnetic field lines.

A new concept - Ampere's Law - let's us tackle nastier problems with relative ease.

We will now discuss the fundamental relationship between magnetic field and current, known as Ampere's Law. This law is more fundamental then the *Biot-Savart* Law (in fact, in an advanced course you might have to derive the BSL from Ampere's Law).

There is an essential piece we need first: the line integral.

The Line Integral

Let's begin this lecture with a discussion of *the line integral*. Let's consider a simple path, one that closes on itself. The simplest such closed path is a circle whose radius is R. The length of the path is just the circumference of the circle: $2\pi R$.

Let's divide the path into equally sized tiny pieces, and describe each piece by vectors, $d\vec{s}$, which are vectors that each point in a direction tangent to their section of the path (little pieces of arclength). The vectors describe the direction and distance you would walk if you were traveling this path in a complete loop.

The line integral is just a way of saying "add up something along the path." We denote it with a special symbol:

$$\oint$$

where the little circle reminds us that we are supposed to sum up all the little pieces around a closed path.

Let's try one of these. For instance, what if we integrate all the pieces of our circular path around the path:

$$\oint ds = 2\pi R$$

We expect this to give us the circumference - the length of the path around the circle.

The path integral of a magnetic field

Let's consider a wire carrying current I. Let's have the current point out of the page toward the viewer (positive z direction). Let's then consider a circular path around the wire, centered on the wire with radius r_1.

If we take a small piece of this path, described by a vector $d\vec{s}$, we can then take the product of the magnetic field on that piece of the path and that vector:

$$\vec{B} \cdot d\vec{s}$$

We know that the magnetic field also circulates around the wire, just like our path, and in the same direction. Thus:

$$\vec{B} \cdot d\vec{s} = B\,ds\,\cos(0) = B\,ds$$

If we consider any piece of the path and consider this product on that piece, as long as the pieces are all at the same distance from the wire and so the magnetic field strength is a constant. Let's then do the path integral:

$$\oint B\,ds = B\oint ds = B(2\pi r_1)$$

Recall the mathematical form for the magnetic field around an infinite wire, a radius r_1 from the wire:

$$B = \frac{\mu_0 I}{2\pi r_1}$$

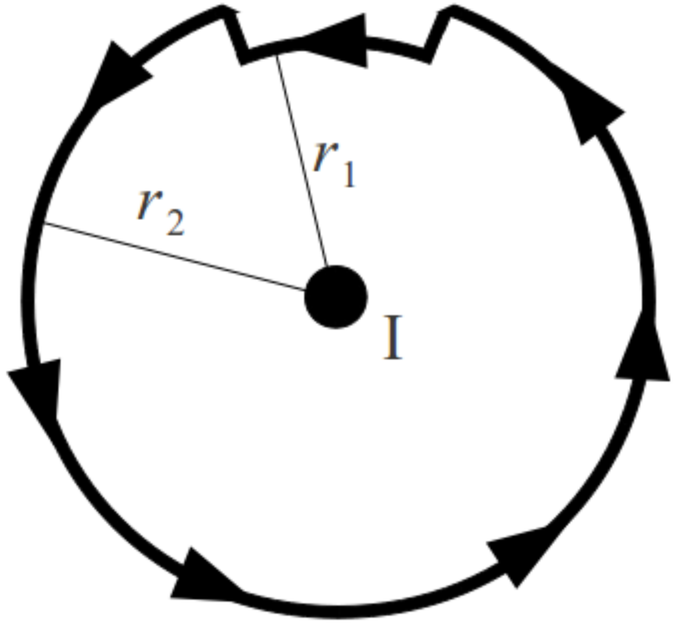
We can plug that into our line integral:

$$\oint B\,ds = B(2\pi r_1) = \frac{\mu_0 I}{2\pi r_1}(2\pi r_1) = \mu_0 I$$

Huh. That's a pretty interesting result. This line integral ends up being *independent of the distance from the wire* and *dependent only on the current enclosed by the path*. Try another circular path with radius $r_2 > r_1$. You'll find the same thing:

$$\oint B\,ds = B(2\pi r_2) = \frac{\mu_0 I}{2\pi r_2}(2\pi r_2) = \mu_0 I$$

In fact, any arbitrary closed path around the wire will lead to the same result - it doesn't have to be a simple circular path. This is seen by considering a circular path that changes radius at some point, then returns to the original radius:



We can write this path integral in four pieces, but along two of them the dot product is zero because the path is perpendicular to the magnetic field. So those little diversions don't count and we get the same result. In fact, *it is universally true that:*

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

This is known as Ampere's Law. Let's explore it further.

Goals

- Demonstrate the application of Ampere's Law
- Discuss the magnetic field due to a solenoid, and its applications

Application: Solar Currents

Show the movie of the solar flares again.

Solar storms consist of large currents of electrically charged particles moving in the sun's magnetic fields. Consider a rectangular Amperian Loop that has long dimension $L = 400 \times 10^9 \mathrm{m}$ and is in the presence of a constant magnetic field whose strength is $B = 2 \times 10^{-3} \mathrm{T}$. What is the total current enclosed by the loop?

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I$$

According to our picture (Wolfson Figure 26.30b), two legs of the loop lie in the direction of the magnetic field, so $\vec{B} \cdot d\vec{r} = Bdr$. For the other two, the dot product is zero. Thus:

$$\oint \vec{B} \cdot d\vec{r} = 2 \int Bdr = 2B \int dr = 2BL$$

This is then equal to:

$$2BL = \mu_0 I$$

We can then solve for the current:

$$I = 2\frac{BL}{\mu_0} = 2\,\frac{(2.0 \times 10^{-3}\mathrm{T})(400 \times 10^9 \mathrm{m})}{(4\pi \times 10^{-7}\mathrm{N/A^2})} = 10^{12}\mathrm{A}$$

Conceptual Puzzle: Direction of Currents in Wired

Ampere's Law is also useful for solving for the direction of unknown currents in wires. Consider three parallel wires, labeled A, B, and C, in which equal magnitude current flows in the same direction in two of them but in the opposite direction in the third (you don't know which is which at first). Consider two loops, one which encircles A and B (Loop 1) and one which encircles B and C (Loop 2). If the following is true:

$$\oint \vec{B} \cdot d\vec{r} = 0 \text{ (Loop 2)}$$

Then:

1. What is $\oint \vec{B} \cdot d\vec{r}$ around Loop 1?

   ANSWER

   If the above is true about Loop 2, it means that the net current enclosed is NONZERO and current in B and C must flow in the same direction. That means that $\oint \vec{B} \cdot d\vec{r} = 0$ (Loop 1) because current must be opposite in one of the three wires to the flow of the other two, and that means current enclosed would be ZERO.

2. Which wire carries the current opposite the other two?

   SOLUTION

   As argued above, it must be wire A.

Application: Magnetic Field in a Long Wire

Consider an infinite straight wire along the x-axis. We've previously treated the wire as thin and calculated the magnetic field outside the wire. We found it to be:

$$B_{outside} = \frac{\mu_0 I}{2\pi y}$$

where y is the distance from the wire.

Let's consider a long, straight wire carrying a constant current, I. The wire now has a thickness; let's treat is as a cylinder whose radius is R. We recognize that this problem has *line symmetry*. This means that we expect that no matter what, the magnetic field will depend only on the radial distance from the center of the wire (the cylinder axis).

Outside the wire, we expect the magnetic field lines to circle around the wire. We can choose our Amperian loops to take advantage of this fact, and make them also circles centered on the wire axis. What about inside the wire? If we draw a circular Amperian loop inside the wire, it will enclose a fraction of the current but that current will still be a cylindrical flow with circular magnetic field lines surrounding it. This in all cases in this problem:

$$\oint \vec{B} \cdot d\vec{\ell} = \oint B d\ell = B(2\pi r)$$

where r is the radius of the enclosed cylinder.

Let's consider the two big cases in this problem: outside the wire, where the enclosed current is fixed and equal to I, and inside the loop where the enclosed current is a fraction of the total current (to be determined).

Outside the wire

Here, the enclosed current is I no matter how far we are from the wire. Thus:

$$2\pi r B = \mu_0 I$$

and thus:

$$B = \frac{\mu_0 I}{2\pi r}$$

where $r = y$, our distance from the center of the wire. We've simply recovered the result from the *Biot-Savart* Law!

Inside the wire

Inside the wire, our Amperian Loops only enclose a fraction of the total current. How do we figure it out?

If the current, I, is constant and uniformly distributed through the wire, then the *current density* is constant in the wire. Thus:

$$J = I/A = I/(\pi R^2)$$

for the whole wire is the same as:

$$J = J_{enclosed} = I_{enclosed}/(\pi r^2)$$

for a current enclosed in a loop whose radius is r. Thus:

$$J = J_{enclosed} \longrightarrow I_{enclosed} = I\frac{r^2}{R^2}$$

Thus:

$$2\pi r B_{inside} = \mu_0 I \frac{r^2}{R^2}$$

and thus:

$$B_{inside} = \frac{\mu_0 I r}{2\pi R^2}$$

In both cases, application of the right-hand rule tells us that the magnetic fields circulate counter-clockwise when viewing the current coming at us.

The Solenoid

One of the most important applications of a current-carrying wire is the solenoid. Take a straight wire and start winding it around a rod or a tube to make a tight coil. Remove the rod/tube. The remaining current-carrying coil generates a magnetic field that is nearly uniform inside the coil and weak outside the coil.

Why is this useful? Many applications need a uniform, well-characterized magnetic field (just like applications like capacitors and particle accelerators benefit from a uniform electric field). Consider MRI machines, where the patient needs to be immersed in a strong, uniform magnetic field. They are actually being inserted into a large solenoid.

Let's analyze the solenoid as an extension of the current loop. Consider the magnetic dipole field resulting from a single current loop. Now add a second loop next to it, then a third. inside the coil, the field becomes more uniform. Outside it grows weaker and weaker. For an *infinitely long solenoid*, the field inside will be perfectly uniform and the field outside will be zero.

Real solenoids are not perfect,, and do have weak an non-uniform electric fields outside the coil. We can analyze the magnetic field of a solenoid very close to the center of a long coil, so that we are far from the ends and the coil is effectively infinite in either direction away from the center.

Draw a cross-section of a coil, carrying a current I. Now draw a rectangular Amperian Loop whose long dimension is L that encloses N turns for the coil. For every additional coil enclosed, the Amperian Loop encloses an additional I of current. Thus the loop will enclose a current NI, and the right-hand side of Ampere's law will be $\mu_0 NI$.

What about the left side? We need to evaluate::

$$\oint \vec{B} \cdot d\vec{\ell}$$

The do product is zero on the top leg of the loop because the field is zero. On the right and left legs, it's zero because the dot product is zero. The only non-zero contribution is on the bottom leg, where the magnetic field is uniform. Thus:

$$\oint \vec{B} \cdot d\vec{l} = B \int_0^L d\ell BL = \mu_0 NI$$

Thus:

$$B = \frac{\mu_0 N I}{L}$$

We can rewrite this in a more generic way by substituting the number of turns of the coil per unit length of the coil, $n = N/L$:

$$B_{solenoid} = \mu_0 n I$$

Even though real solenoifs are not infinitely lng, this formula gives a very good approximation of the field inside a real solenoid.

MRI Solenoid

A typical bore MRI scanner (made from a large solenoid) can generate a magnetic field of 1.5T. The coil wire is superconducting, and can easily carry hundred of Amps of current. Let's assume a typical MRI scanner carries 300A of current. How many turns per meter are in the solenoid?

$$n = B/(\mu_0 I) = (1.5\mathrm{T})/\left((4\pi \times 10^{-7}\mathrm{N/A^2})(300\mathrm{A})\right) \approx 4000 \text{ turns per meter}$$