Modern Physics (PHY 3305) Lecture Notes

HomeworkSolutions005

SteveSekula, 16 March 2010 (created 15 March 2010)

no tags

Point Distributions

Points were distributed as follows for each problem:

Problem	Total	Point Distribution
CH5-1	5	Recognized "small" is needed for wave behavior to manifest (2 Points) and "bound" is needed to constrain the behavior of the wave (3 Points)
CH5-2	10	Noted the importance of the wave nature of the electron in an atom (2 Points), notes it was bound in that space (4 Points), and that the binding has consequences both for the number of energy states and the lowest energy state (4 Points)
CH5-4	15	Noted that transitions between energy levels in atoms causes photons to be emitted (5 Points), noted that only certain transitions are possible (5 Points), and that this results in the observation of distinct colors on the CD (5 Points)
CH5-7	10	Appled the Normalization Requirement to solve the problem (10 Points)
CH5-21	20	Part a: Explained the limiting behavior (3 Points) and sketched the function (1 Point); Parts b and c: Applied the definition of a turning point (4 Points) to write the equation for those positions, and solved for them (4 Points)
CH5-23	20	Correctly solved for all the parts of the wave function (15 Points) and placed units next to all numerical results in the equation (5 Points)
CH5-26	20	Treated problem as infinite square well (10 Points), recognized that differences in energy levels yield the photons from transitions (5 Points), and computed the energy of a transition (5 Points)
CH5-40	50	Divided the problem into regions (15 Points), solved for the wave functions in each region (25 Points), and used the boundary conditions to obtain the quantization condition (10 Points)
CH5-94	10	Drew the function (5 Points), argued as to whether this was the ground state (3 Points), and reasoned correctly that it likely is not (2 Points)
CH5-95	20	Defined "most probable" by the condition on the first derivative (10 Points) and computed the most probable locations (10 Points). If you instead tried to compute the expectation value, you'll lose 2 points for confusing "average" with "most probable".
CH5-96	20	Wrote the Schroedinger Wave Equation for this situation (10 Points) and solved the equation for U(x) (10 Points)

Deductions outside of the above:

- 1 point is deducted for failing to box the numerical answer
- 1 point is deducted for incorrectly applying significant figures
- other points are deducted as outlined in the homework policy

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- HARRIS, *CH5-1* (5 Points)

SOLUTION

"Small" is required because the wave nature of the particle needs to be apparent, and for that to be true the relevant dimensions of the situation must be comparable to the wavelength. "Bound" is required because quantization arises when the wave function is constrained by boundaries in the problem such that only specific wavelengths/frequencies/energies are possible for the particle in the situation.

- HARRIS, *CH5-2* (10 Points)

SOLUTION

On the scale of an atom, the electron is not a "point particle." Rather, its wave nature is fully apparent. As such, in order to remain confined around the nucleus the electron wave must behave in such a way that physicality is maintained - for instance, there cannot be discontinuities in the wave function or its first derivative, otherwise infinite energies are possible. Thus, the wave is forced to admit only certain frequencies, which answers the first complaint. In addition, there can be a minimum frequency which is non-zero. An electron can thus never totally radiate away all of its energy. This statement answers the second complaint. The key here is to recognize that the electron is not a point particle, and its "physical" behavior is thus governed by a different set of constraints and expectations.

- HARRIS, *CH5-4* (15 Points)

SOLUTION

A fluorescent light works because a gas is ionized. Since an atom in the gas consists of electrons which can be excited to various energy levels, only to emit photons when they drop back down to the ground state, such lights emit a spectrum which consistent of a large number of discrete wavelengths closely spaced (so as to appear "white" to the eye). As a result, if you scatter the light off of a grated surface, such as the surface of a CD, different wavelengths will scatter at different angles and your eye will separate them. This simple experiment reveals immediately the quantum character of the atom; the eye will perceive distinct color bands in the scattered light, rather than a continuous rainbow of color.

- HARRIS, *CH5-7* (10 Points)

SOLUTION

The $\psi(x)$ in equation (5-16) is

$$\psi(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$$

when $0 < x < L$ and is zero everywhere else. Let us consider the case where $n = 0$. In that case, $\psi(x)$ is zero EVERYWHERE. However, since physicality requires that

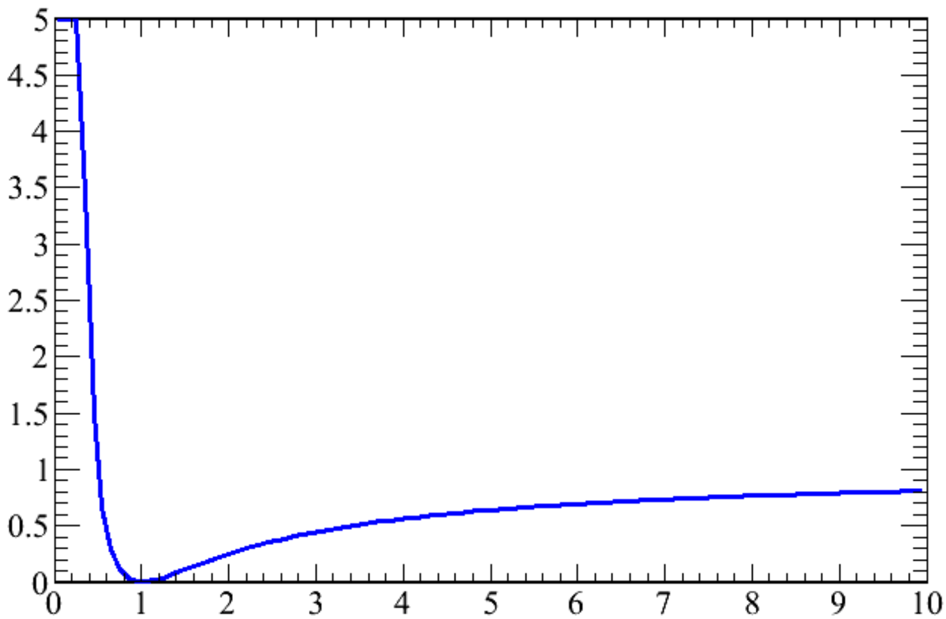
$$\int |\psi(x)|^2 dx = 1,$$

we cannot have a situation where the wave function is zero everywhere (since this integral is zero for $n = 0$).

- HARRIS, *CH5-21* (20 Points)

SOLUTION

Part (a): Let us determine the limiting behavior first. When $x = 0$, $U(x) = \infty$. When $x = \infty$, $U(x) = 1$. Now, let us consider where $U(x)$ is minimal. To do this, we need to determine at what x the first derivative is zero. Computing this, we find that when $x = 1$ the first derivative is zero. We can then plot the function and see if this all holds:



Part (b): Assume that E = 0.5J. We have to find turning points - places where the total energy of the particle is equal to the potential energy (find x such that $E = U(x)$. If we write the equation as:

$$\frac{1}{2} = 1/x^2 - 2/x + 1$$

we can then re-write this as a quadratic equation,

$$x^2 - 4x + 2 = 0$$

and solve for x. There are two real-valued x locations, $x = 2 \pm \sqrt{2} = 3.41$ and 0.59. We can see from the plot of the function that when $E = 0.5$ we DO expect two turning points.

Part (c): Repeating the same exercise as in (b), we can then re-write the requirement that $E = U(x)$ as a quadratic equation,

$$x^2 + 2x - 1 = 0$$

and solve. The only physical turning point on x (only real-valued location on x) is $x = 2\sqrt{2} - 1 = 1.82$. We can see from the plot of the function that for this large an energy, we only expect a turning point on the left-side of the function; to the right, the motion is unbound.

- HARRIS, *CH5-23* (20 Points)

SOLUTION

The wave function for the $n = 3$ state looks like this

$$\psi_3(x) = \sqrt{2/L}\sin(3\pi x/L)e^{-i(E/\hbar)t}$$

for $0 < x < L$ and is zero everywhere else. Concentrate only on the non-zero part. The energy of this state is given by $E = n^2\hbar^2\pi^2/(2mL^2)$. The well is 10nm wide ($L = 10nm$), and from that we can compute the energy to be $E = 5.43 \times 10^{-21}J$. We can now write the wave function with only x and t unspecified:

$$\psi_3(x) = (1.4 \times 10^4 \mathrm{m}^{-\frac{1}{2}}) \sin\left((9.4 \times 10^8)\mathrm{m}^{-1} \cdot x\right) e^{-i(5.2\times 10^{13})\mathrm{Hz}\cdot t}.$$

- HARRIS, *CH5-26* (20 Points)

Let us treat the nucleus as an infinite square-well and see how much progress we can make. When "nucleons drop to lower energy levels," they are necessarily transitioning from a higher-energy state to a lower-energy state. Let us consider the case where a nucleon in the $n = 2$ state drops to the $n = 1$ state. The energy difference will represent the energy of the photon emitted in the transition. Computing this, we find:

$$\Delta E_{2,1} = (2^2 - 1^2)\frac{\hbar^2\pi^2}{2mL^2}.$$

The mass of the nucleon is the mass of either a proton or neutron (about 1.673×10^{-27}kg). The size of the well can be taken to be the size of a nucleus - about 15×10^{-15}m. We have all of the ingredients to calculate the energy difference:

$$\Delta E_{2,1} = 4.38 \times 10^{-13}\mathrm{J} = 2.74\mathrm{MeV}.$$

So in this example we see a "typical" transition energy of a few *MeV*, which is consistent with the observation that photon emitted from nuclear transitions are in this range.

- HARRIS, *CH5-40* (50 Points)

We can break the problem into three regions: Region I is the "well" ($0 < x < L$), Region II is the infinite wall ($x \leq 0$), and Region III is the finite wall ($x \geq L$). Let us then consider the form of the wave functions in each region:

- Region II: let's begin here, as this is the easiest. The wave function must VANISH in this region, so $\psi(x) = 0$ for $x \leq 0$.
- Region I: The left side of the well is an infinite wall, below which the wave function must vanish. A good function that satisfies that requirement is $\psi(x) = A\sin(kx)$
- Region III: The wave function is not required to vanish in this region as it was in Region II. We can guess that the function probably has a form like $\psi(x) = Fe^{-fx}$in this region, to prevent the function from running off to infinity as $x \to \infty$.

Let us begin by solving for k in Region I using the Shroedinger Wave Equation. Plugging this solution into the wave equation in that region,

$$-\hbar^2/2m\,\frac{d^2\psi(x)}{ddx^2} = E\psi(x)$$

yields $k = \sqrt{2mE}/\hbar$.

Then, let us solve for f in Region III. To obtain this, we plug the wave function into the Shroedinger Wave Equation in this region:

$$-\hbar^2/2m\,\frac{d^2\psi(x)}{ddx^2} + U_0\psi(x) = E\psi(x)$$

$$\frac{d^2\psi(x)}{ddx^2} = \frac{2m(U_0 - E)}{\hbar^2}\psi(x).$$

Plugging in our test wave function for this region, we obtain the result that

$$f = \sqrt{\frac{2m(U_0 - E)}{\hbar}}.$$

We can now use the boundary conditions (continuity) to work on the remaining coefficients. Let us focus on the Region I/Region III boundary, which is much more unconstrained than the other boundary. We impose our continuity constraints:

- Continuity of the wave function: $\psi_I(x) = \psi_{III}(x)$when $x = L$, or

$$Asin(kL) = Fe^{-\sqrt{\frac{2m(U_0-E)}{\hbar^2}}L}$$

  - Continuity of the first derivative of the wave function:

$$\left.\frac{d\psi_I(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}(x)}{dx}\right|_{x=L}$$

  or

$$Ak\cos(kL) = -F\cdot\sqrt{\frac{2m(U_0 - E)}{\hbar}}e^{-\sqrt{\frac{2m(U_0-E)}{\hbar}}L}.$$

  We have two equations and two unknowns, but let us forget about solving for the coefficients and just go for the quantization condition. If we take the ratio of the equations obtained by continuity requirements, we get a single equation with no unknown coefficients:

$$\frac{Ak\cos(kL)}{A\sin(kL)} = \frac{-F \cdot \sqrt{\frac{2m(U_0-E)}{\hbar}}e^{-\sqrt{\frac{2m(U_0-E)}{\hbar}}L}}{Fe^{-\sqrt{\frac{2m(U_0-E)}{\hbar}}L}}$$

$$\cot(kL) = -\sqrt{\frac{2m(U_0-E)}{\hbar k}}.$$

If we then substitute for $k = \sqrt{2mE}/\hbar$ on the right-hand side of this equation, we obtain

$$\cot(kL) = -\sqrt{U_0 - E}/\sqrt{E}.$$

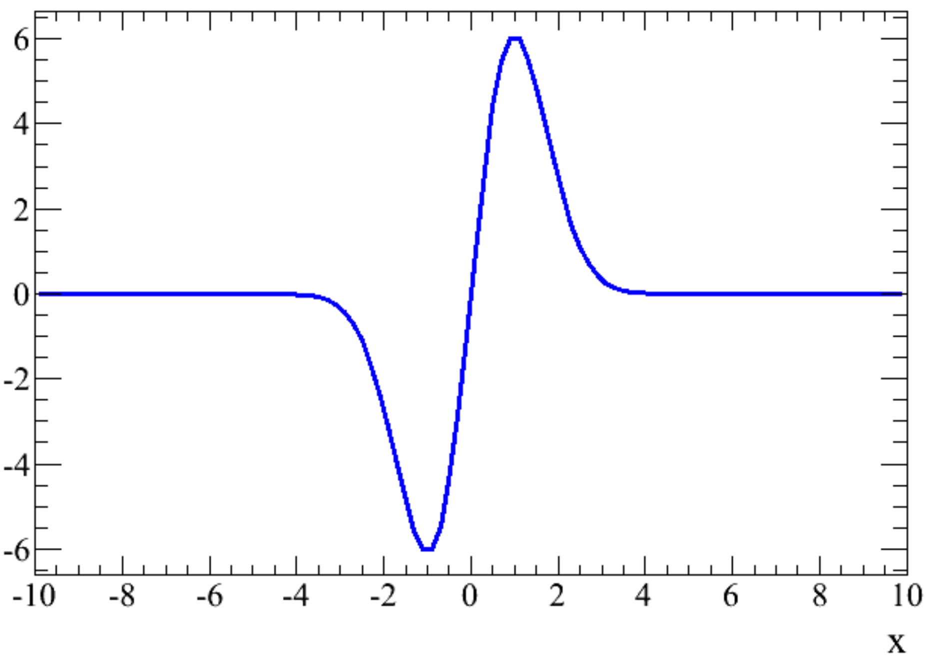
This then yields the quantization condition,

$$-\sqrt{E}\cot(kL) = \sqrt{U_0 - E}.$$

- HARRIS, *CH5-94* (10 Points)

SOLUTION

Let us begin by drawing the wave function, using $A = 10$ and $b = 1$:



The function has three nodes (places where it goes to zero: $-\infty$, 0, and $+\infty$. This is unlikely to be the ground-state wave function of the system, since it is possible to have a wave function with only two nodes (e.g. $\psi(x) = Ae^{-x^2/2b}$).

- HARRIS, *CH5-95* (20 Points)

SOLUTION

The *most probable* location to find the particle is the place where the probability density is largest - that is, maximal. This is different from the expectation value, which is where the particle will be, on average, in x. In this case, if you compute the expectation value you obtain zero, which (based on the plot above) makes sense - since the probability density, $|\psi(x)|^2$ will be symmetric around zero, on average the particle will be found at zero. Another way to see this is that the product of $x|\psi(x)|^2$ is an ODD function, one where $f(x)_{x>0} = -f(x)_{x<0}$(or, as is usually written, $f(-x) = -f(x)$) such that the sum of the integrals above and below zero must necessarily equal zero. You can see this as follows:

$$\int_{-\infty}^{\infty} x^3 e^{-x^2} dx = \int_{-\infty}^{0} x^3 e^{-x^2} dx + \int_{0}^{\infty} x^3 e^{-x^2} dx = \int_{0}^{\infty} (-x)^3 e^{-(-x)^2} dx + \int_{0}^{\infty} x^3 e^{-x^2} dx = -\int_{0}^{\infty} x^3 e^{-x^2} dx + \int_{0}^{\infty} x^3 e^{-x^2} dx = 0.$$

In this problem, however, we are concerned with where the probability density is maximal. Let us write the wave function as

$$\psi(x) = Axe^{-\frac{1}{2}ax^2}$$

, where $a = 1/b^2$. To find the place of maximal probability density, we need to find the location(s) along x where the first derivative of the probability density is zero:

$$\frac{d}{dx} A^2 x^2 e^{-ax^2} = 0.$$

Taking the derivative yields:

$$A^2 x e^{-ax^2} \left(2 - 2ax^2\right) = 0.$$

So we don't have to solve for A to get the answer to this problem, since it drops out! We can then solve for x under this condition, and we find $x = \pm 1/a = \pm b$.

- HARRIS, *CH5-96* (20 Points)

SOLUTION

We can write the Schroedinger Wave Equation for the case where $E = 0$:

$$-\hbar^2/2m \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = 0.$$

We can the insert our wave function and obtain:

$$-\left(\hbar^2/2m\right) Axe^{-x^2/2b^2} \left(-1/b^2 - 2/b^2 + x^2/b^4\right) = -U(x)Axe^{-x^2/2b^2}.$$

Equating terms on each side, we find

$$U(x) = \left(-\frac{3\hbar^2}{2mb^2} + \frac{\hbar^2 x^2}{2mb^4}\right).$$