Modern Physics (PHY 3305) <small>Lecture Notes</small>

HomeworkSolutions006

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Point Distributions

no tags

Points were distributed as follows for each problem:

Problem	Total	Point Distribution
CH6-1	5	Recognized that a bound state means "wave function is zero at infinity" and that boundary conditions in this problem cannot guarantee this condition (3 Points). Argued based on the understanding of bound states whether or not this is a bound state (2 Points)
CH6-2	5	Applied understanding of barrier tunneling in quantum physics (3 Points) to argue which energies represent the possibility of being found infinitely far to the right in quantum (1 Point) and classical (1 Point) mechanics.
SS-9	40	Part 1: recognized that this is a tunneling problem through a wide barrier and applied that transmission probability to solve the problem (10 Points). Solved for the breaking force from the pressure and area of the ice cube (2 Points), and solved for the height of the potential barrier represented by this force (3 Points). Solved for the kinetic energy of the ice cube and computed the transmission probability (5 Points). Part 2: Recognized that a 50% transmission probability cannot be accomplished with a wide barrier, and applied the general transmission probability for a barrier (10 Points). Inverted the equation and solved for $\hbar$ (10 Points)
CH8-25	20	Part a: recognized that the angular momentum of the electron at rest is given by the spin (5 Points). Recognized that the relativistic momentum form for p was the correct expression to insert for p (3 Points), and solved for γ or the speed (2 Points) Part b: Recognized that the total internal energy of an electron is given by its rest mass (5 Points), and

		compared rest energy and literal spin energy (5 Points).
CH8-35	10	Correctly setup the integrals for the probability of the symmetric and antisymmetric states (8 Points) and then calculated the energies in the two cases (2 Points).
CH8-41	20	Recognized that the principle quantum number determined the total energy of each level (4 Points) while spin allows for different numbers of particles to occupy a given level (10 Points). Computed the energy of the system for each case (6 Points).

Deductions outside of the above:

- 1 point is deducted for failing to box the numerical answer
- 1 point is deducted for incorrectly applying significant figures
- other points are deducted as outlined in the homework policy

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HARRIS *CH6-1* (5 Points)

SOLUTION

The situation depicted is unlikely to represent a bound state. A "bound state" is one where the motion of a particle is constrained to a region by an external force in such a way that the wave function goes to zero at infinity (probability of finding the particle infinitely far from the well is zero). Boundary conditions on the wave functions in the 5 regions indicated will should significant presence of the wave function in the regions outside the walled box. Once outside the walled region, the particle wave function is unconstrained on one side, and thus the particle is free once it tunnels through the box walls. Since there are places for $E < U_0$ where the particle can achieve freedom (non-zero probability of being found at infinity), this is not a bound state.

HARRIS *CH6-2* (5 Points)

SOLUTION

For quantum particles whose behavior is defined by probability waves, the situation where $E > U_1$ represents one where there is a possibility of being found infinitely far to the right; for $E < U_1$, the wave function at $+\infty$ is zero by boundary conditions on the wave function. When $E > U_1$, the wave function can

be non-zero at $+\infty$. If the particle instead were purely classical, then $E > U_2$ is necessary before the particle can be found at $+\infty$. This is because classical particles can be treated as definite objects, not waves, and thus cannot be found in "classically forbidden" regions.

Problem *SS-9* (40 Points)

1. While on Spring Break, you find yourself in a sunny location with time to kill (relaxation is SO boring). You are enjoying a university-sanctioned, non-alcoholic, fruity drink, cooled in the hot sun by an ice cube. One of the ice cubes, moving at a leisurely pace of about 1.0 cm/s, strikes a wall of the glass. When the flat side of the ice cube clinks against the flat side of the glass, a thought occurs to you. Stopping one of the passing waiters, you ask, "Excuse me, but what is the weight of your ice cubes?" "Why," he replies, "we pride ourselves on the fact that every ice cube is, in fact, a perfect cube (3.0cm on a side) weighing EXACTLY 27.0 grams." "And what is the breaking strength of your drinking glasses?" The waiter seems puzzled as to why this matters. "Why, I checked the company's specifications this morning and I can state unequivocally that each glass, whose walls are 0.50cm thick, requires 50.0 mega-pascals of pressure before it will break completely." You use this information to determine the probability that an ice cube will tunnel through the glass and appear unharmed on the table next to the glass. What probability did you find? *HINT: Simplify the problem by treating the ice cube as a single particle. HINT: compute the penetration depth of the ice cube wave function into the glass and determine if the glass represents a thin or a thick barrier. HINT: You can treat the inside and outside of the glass as "zero-potential" regions but you need to compute the potential barrier represented by the glass wall, remembering that $U = F \cdot \Delta x$. HINT: you will encounter some "interestingly sized" numbers in this problem. Feel free to take advantage of the fact that $ln(Ae^B) = ln(A) + B$ to quote your answer.*
2. How big would Planck's Constant have to be to make the probability of passing through the glass wall be 50.0%?

SOLUTION

This problem is a tunneling problem. You need to determine how likely it is for the ice cube, treated as a single quantum particle striking a barrier, to be found on the other side of the barrier. In order to solve Part 1 of this problem, you need to know the potential represented by the glass wall, then apply the tunneling transmission probability calculation to solve the problem. For Part 2, the transmission probability is known and you have to invert the transmission probability formula to solve for Planck's constant.

Solution to Part 1:

Let's begin by computing the kinetic energy of the ice cube and the potential barrier represented by the wall. The kinetic energy is given by the non-relativistic form, since the velocity is so much smaller than that of light. This is $E = \frac{1}{2}m\nu^2 = 1.35 \times 10^{-6}J$. The potential barrier is a little trickier. In order to determine the potential, you have to use $U = F_{breaking} \times \Delta x$, where the potential is the product of the breaking force (force required to overcome the barrier - that is, "smash the glass") and the thickness of the glass. The breaking force is related to the maximum pressure the glass can withstand, so we need to know what force the ice cube would have to exert in order to break the glass. $F_{breaking} = P_{breaking} \cdot A_{cube} = (50 \times 10^6 N/m^2) \cdot (9.0 \times 10^{-4}m^2) = 45,000N$. We can now compute the potential represented by the glass:
$U = (45,000N) \cdot (0.005m) = 225\text{N} \cdot \text{m}$

We can already see that $U >> E$. Let's apply that to determine whether we can simplify our treatment of the barrier - is the barrier wide? To determine that, we much compute:

$$L/\delta = \frac{\sqrt{2,(U_0 - E)}}{\hbar}L = 1.65 \times 10^{32}$$

which is significantly great than 1. The barrier is wide.

With that in mind, we can proceed with the rest of the computation. The wide barrier tunneling probability is

$$T \approx 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0 - E)}/\hbar}.$$

Let's compute various parts of this:

$$16\frac{E}{U_0} = 9.6 \times 10^{-8}$$

$$(1 - E/U_0) = 1 - 6 \times 10^{-9}$$

$$-2L\sqrt{2m(U_0 - E)}/\hbar = -3.3 \times 10^{32}$$

The last of these numbers is so small that in order to get any meaningful reduction of the formula to a simpler form, we can instead compute $ln(T)$:

$$ln(T) \approx ln(16\frac{E}{U_0}) + ln\left(1 - \frac{E}{U_0}\right) - 2L\sqrt{2m(U_0 - E)}/\hbar = -16.16 - 6.0 \times 10^{-9} - 3.3 \times 10^{32} \approx -3.3$$

since the first two terms in the sum are totally negligible compared to the last. Thus we arrive at:

$$ln(T) \approx -3.3 \times 10^{32}$$

and we see that the chance of an ice cube tunneling all at once, unharmed, through a glass wall is (for all practical purposes) completely negligible.

Solution to Part 2:

When the probability of tunneling is 50%, the barrier is no longer wide. A large tunneling probability means the barrier is entirely surmountable (and quickly), so if we are to solve for Planck's constant we have to begin with the full-blown tunneling formula when $E < U_0$:

$$T = \frac{4(E/U_0)(1 - E/U_0)}{\sinh^2\left[\sqrt{2m(U_0 - E)}L/\hbar\right] + 4(E/U_0)(1 - E/U_0)}.$$

There is a term common to the numerator and the denominator. Let is leadingly compute the following:

$$\mathcal{E} = 4(E/U_0)(1 - E/U_0) = 2.4 \times 10^{-8}.$$

Let us solve for $\hbar$:

$$1/\hbar = \frac{1}{\sqrt{2m(U_0 - E)}L}\operatorname{arcsinh}\left(\frac{\mathcal{E}}{T} - \mathcal{E}\right)^{\frac{1}{2}}.$$

We can write the arcsinh as

$$\operatorname{arcsinh} = ln(x + \sqrt{x^2 + 1}) = 1.5 \times 10^{-4}.$$

From this, we can solve for $\hbar$:

$$\hbar = 112.5 \text{ J} \cdot \text{s}.$$

This then yields

$$h = 2\pi\hbar = 706.9 \text{ J} \cdot \text{s}.$$

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HARRIS *CH8-25* (20 Points)

SOLUTION

Part (a):

Treat the electron as a tiny spherical shell whose mass is entirely spread over

the shell at radius $r = 10^{-18}$ m. The spin (intrinsic) angular momentum of the electron is $S = \sqrt{s(s+1)}\hbar = \sqrt{1/2(1/2+1)}\hbar = 9.14 \times 10^{-35}$ J·s. We can relate this angular momentum to the speed the shell would have to be rotating via $S = rp$ and $p = \gamma m u$. The relationship (given on the mid-term) $\gamma u = c\sqrt{\gamma^2 + 1}$ will be handy. From this, we can solve for the gamma factor and thus the speed:

$$\gamma = \sqrt{\left(\frac{S}{rmc}\right)^2 - 1} = 3.35 \times 10^5$$

which is HIGHLY relativistic.

Part (b):

The known internal energy of the electron is given entirely by its rest mass,

$$E_{internal} = mc^2 = 8.19 \times 10^{-14} \text{ J.}$$

The internal energy represented by this rotation is given by

$$E \approx pc = \left(9.14 \times 10^{-17} \text{ kg·m/s}\right) \cdot \left(2.998 \times 10^8 \text{ m/s}\right) = \left(2.74 \times 10^{-8}\right) \text{ J.}$$

We see that the internal energy of the electron, sitting at rest and actually spinning this fast, is six orders of magnitude larger than the known internal energy (rest mass) of the electron. Therefore, it is completely incorrect to think of "spin" as actual motion and not simply an intrinsic and irreducible property of a particle.

HARRIS *CH8-35* (10 Points)

The states occupied by the particles in the box are $n = 1$ and $n' = 2$. The corresponding wave functions are given in Example 8.2:

$$\psi_S(x_1, x_2) = \frac{\sqrt{2}}{L}\left(\sin\frac{1\pi x_1}{L}\sin\frac{2\pi x_1}{L} + \sin\frac{2\pi x_1}{L}\sin\frac{1\pi x_1}{L}\right)$$

$$\psi_A(x_1, x_2) = \frac{\sqrt{2}}{L}\left(\sin\frac{1\pi x_1}{L}\sin\frac{2\pi x_1}{L} - \sin\frac{2\pi x_1}{L}\sin\frac{1\pi x_1}{L}\right)$$

where the two energy levels have been used in writing the wave functions. Following example 8.2, we can write the probability integral,

$$P(left) = \int_0^{L/2} |\psi(x_1, x_2)|^2 dx_1 dx_2$$

as

$$P(left) = 2/L^2 \int_0^{L/2} \sin^2 \frac{1\pi x_1}{L} dx_1 \int_0^{L/2} \sin^2 \frac{2\pi x_2}{L} dx_2 + 2/L^2 \int_0^{L/2} \sin^2 \frac{2\pi x_1}{L} dx_1 \int_0^{L/2} \sin^2 \frac{1\pi x_2}{L} dx_2$$

$$\pm 2 \cdot 2/L^2 \int_0^{L/2} \sin \frac{1\pi x_1}{L} \sin \frac{2\pi x_1}{L} dx_1 \int_0^{L/2} \sin \frac{2\pi x_1}{L} \sin \frac{1\pi x_1}{L} dx_2$$

Doing the integrals, we obtain:

$$P(left) = 2/L^2(L/4)(L/4) + 2/L^2(L/4)(L/4) \pm 4/L^2 \cdot L/2\pi \cdot 4/3 \cdot L/2\pi \cdot 4/3 = 1/4 \pm 0.180$$

So for the symmetric wave function, the probability is 0.43, while for the anti-symmetric wave function the probability is 0.07.

HARRIS *CH8-41* (20 Points)

The "A Closer Look" section on two particles in a 1-D box (page 301) shows that the total energy of the two-particle system is the sum of the energies of the individual particle states. For this problem, we need to determine the states that each of the five particles can occupy in their lowest energy configurations, and then calculate that energy.

SOLUTION

The quantum numbers for our particles in the box will be their principle quantum number, n, which determines the overall energy of the particle, and the z-projection of the particle spin, $m_s = -s, -s+1, ..., s-1, s$, where s is the spin quantum number of the particle. A given quantum state is labeled by (n, m_s).

For the case where all particles are spin-1/2, we cannot put all of the particles in the same quantum states. What quantum states are possible?

- The first particle can go straight to the ground state, $n = 1$, and can have either $m_s = +1/2$ or $m_s = -1/2$. Let's choose that the state it occupies is $(1, +1/2)$.
- The second particle can also go to the ground state, but only if it has opposing spin quantum number. Thus, $(1, -1/2)$.
- The third particle cannot go to the ground state, since it is not filled. It must go to $n = 2$, and can be in state $(2, +1/2)$
- The fourth particle can also be in the n=2 level, with state $(2, -1/2)$
- The last particle cannot be in either of n=1,2, and so must be either of

$(3, \pm 1/2)$.

Since principle quantum number determines total energy, we have our answer:

$$E_{total}^{1/2} = \frac{\pi^2 \hbar^2}{2mL^2}(2 \cdot 1^2 + 2 \cdot 2^2 + 1 \cdot 3^2) = \frac{19\pi^2 \hbar^2}{2mL^2}$$

What about when the spin of the particle is 1? Spin-1 particles are bosons and can ALL occupy the same state, so the lowest energy state occurs when all 5 are in the ground state:

$$E_{total}^{1} = \frac{5\pi^2 \hbar^2}{2mL^2}$$

which is less than the spin-1/2 case.

Finally, we have the spin-3/2 case. Here we again have fermions which cannot all occupy the same state, but thanks to their larger spin there are more spin projections possible, and thus more spin states in the n=1 state. The states are:

- $(1, -3/2)$, $(1, -1/2)$, $(1, +1/2)$, $(1, +3/2)$, $(2, ANYTHING)$ (the last spin can be projected in any way, since it's the only particle in the n=2 level.

The lowest energy state is then:

$$E_{total}^{3/2} = \frac{8\pi^2 \hbar^2}{2mL^2}$$

which is still higher-energy than the boson case, but lower energy than the spin-1/2 case.