Modern Physics (PHY 3305) Lecture Notes

HomeworkSolutions007

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Point Distributions

no tags

Points were distributed as follows for each problem:

Problem	Total	Point Distribution
CH9-2	5	Recognized that macrostates are characterized by large or average properties (3 Points) and that microstates are characterized by the specific properties of molecules (2 Points)
CH9-19	20	Part a: Correctly computed the probability of a 5-particle imbalance (3 Points for the correct number of ways and 3 Points for the correct probabilities). Part b: Correctly computed the probability of a 5% imbalance ((3 Points for the correct number of ways and 3 Points for the correct probabilities). Part c: recognized that a trillion molecule imbalance is small compared to the number of air molecules in a room (4 Points), and that this is a small fluctuation in such a large sample (4 Points)
CH9-35	20	Part a: correctly wrote the formula for the average energy (5 Points) and made a plot of the distribution as a function of temperature (or something related to temperature) (5 Points). Part b: commented on the behavior of the system at low and high T. (5 Points) Part c: argued how this was an appropriate model of paramagnetism (5 Points)
SS-10	25	Part 1: drew the potential well with particles included and labeled (5 Points). Part 2: correctly obtained the occupation numbers of the filled states (5 Points) and recognized that $3E_0$

		had an unfilled state (5 Points) Part 3: Correctly labeled the Fermi energy (5 Points). Part 4: Correctly calculated the work function (5 Points)
CH9-66	30	Part a: Correctly calculated the Fermi velocity (10 Points). Part b: Used the De Broglie wavelength and its relationship to momentum (7 Points) to correctly solve for the wavelength (3 Points). Part c: advanced an argument based on relevant dimensions to explain whether the gas can be treated classically or quantum mechanically (8 Points) and arrived at the correct answer (2 Point)

Deductions outside of the above:

- 1 point is deducted for failing to box the numerical answer
- 1 point is deducted for incorrectly applying significant figures
- other points are deducted as outlined in the homework policy

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HARRIS *CH9-2* (5 Points)

SOLUTION

The *macrostate* of the air in a room can be described by average properties of the entire system, such as temperature, pressure, volume, total number of particles, etc. The *microstate* of the air would be described by the velocity of each individual air molecule, the position of each air molecule, and for a system that much be treated as fully quantum, the quantum state of each particle (all of the applicable quantum numbers).

HARRIS *CH9-19* (20 Points)

SOLUTION

A two-sided room is a *two-state system*. Let us denote the total number of particles as N, and the number of particles on the right side of the room as

N_R. The number of ways that N_R particles can be on the right side of the room is given by $N!/(N_R!(N - N_R)!)$, and the total number of ways of putting some number of particles on the right side of the room is 2^N.

Part a: An imbalance of 5 particles, denoted as $N_R = 1/2\,N + 5$, will be compared for N=20 and N=60. The number of ways that this can happen for N=20 is $20!/15!5! = 15504$. The number of ways to distribute N=20 particles in this two-state system is $2^{20} = 1048576$. Thus the likelihood of a 5-particle imbalance is $15504/1048576 = 0.0148 = 0.01$, or a 1% probability. For N=60 particles, the number of ways to get a 5-particle imbalance is 5.192×10^{16} and the number of ways to distribute is 1.153×10^{18}. The probability of a 5-particle imbalance is then 0.0450, or 5%.

Why is it larger for the room with more particles? 5 particles is a much smaller fraction of 60 then of 20. It should make sense that a smaller *percent* fluctuation is more likely for a big system then for a small system. A 5-particle imbalance for a 20 particle system means 75% of the particles are on one side of the room, while for a 60 particle system it means 58% of the particles are on one side of the room. The latter is more likely in a big system than the former in a small system. For a big system, its easy for a small absolute number of particles to wander to the other side without significantly changing the macrostate.

Let's explore this in part b:

Part b: Which is more likely to have an imbalance of 5%? For N=20, this means $N_R = 11$, while for N=60 this means $N_R = 33$. The probability for N=20 is $167960/1048576 = 0.160 = 0.2$, or 20%. The probability for N=60 the probability is $(8.80 \times 10^{16})/1.15 \times 10^{18}) = 0.076 = 0.08$, or 8%. This confirms the expectation stated at the end of part (a).

Part c: An average size room is likely to have a trillion more molecules on one side than the other (an imbalance of 1×10^9 molecules. Why may we say that precisely half will be on each side? If a room contains Avagadro's number worth of air molecules, 6.022×10^{23}. A trillion is a very tiny percent of such a large number (1.7×10^{-15}, to be exact). As we learned in part (a), a small-percent fluctuation is possible for a large system without changing the macrostate, so by saying "exactly half" of all molecules are one each side of the room we are only wrong by a very tiny fraction.

HARRIS *CH9-35* (20 Points)

SOLUTION

We have a simple thermodynamic system with two states, one with $E = 0$ and $E = E_u$. The latter is the higher-energy state.

Part (a): to compute the average energy with just two states, we can use the exact sum to get the average energy. We'll use the Boltzmann distribution, since we're not told whether these are distinguishable or indistinguishable particles (later is talks about atoms in a solid, which are so large and well spaced as to essentially be distinguishable). The average energy in the exact sum over states is given by:

$$\overline{E} = \sum_{i=1}^{2} E_i P_i(E_i)$$

Applying the Boltzmann probability:

$$\overline{E} = \frac{0e^0 + E_u e^{-E_u/k_B T}}{e^0 + e^{-E_u/k_B T}} = \frac{E_u e^{-E_u/k_B T}}{1 + e^{-E_u/k_B T}}$$
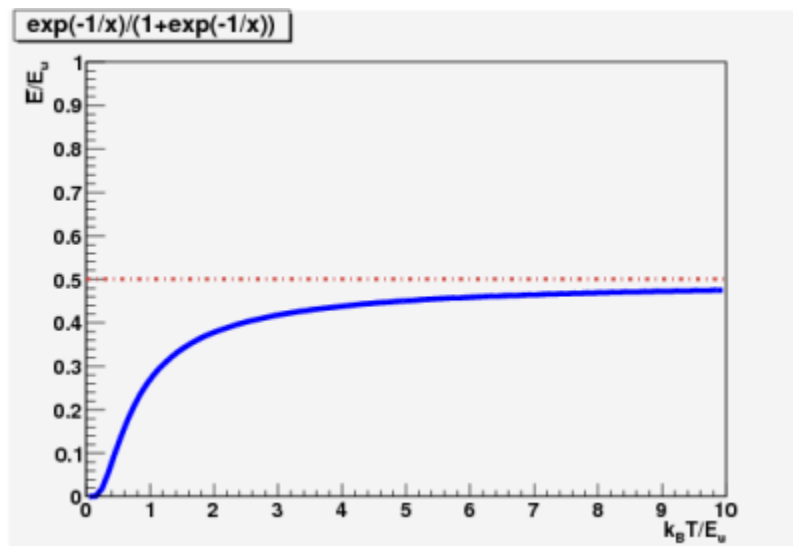
A note on the way I plotted this: a useful trick, especially when you don't know the value of a parameter like E_u, is to absorb the unknown parameter into either the x or y axis (or both!). In the above, you see that I divided both sides of the average energy equation by E_u ($\overline{E}/E_u$), and defined the

x-axis as $x = k_B T / E_u$, which is the inverse of the exponential's argument.

Part (b): Qualitatively, why should this system behave as it does? As $T \to 0$, it makes sense that the system will seek to enter the lowest overall energy macrostate - the one where all atoms are in the $E = 0$ microstate. What about the large-temperature behavior? We see that as $T \to \infty$, the average energy does not tend to E_u but $1/2 \, E_u$ - that is, half the atoms are in the higher energy state and half in the lower energy state. At high temperature, you are just as likely to be in the high as the low-energy state.

Part (c): In a paramagnetic material (c.f. http://en.wikipedia.org /wiki/Paramagnetism), atoms' spins can either align or anti-align with an external magnetic field. When aligned, the energy of an atom is at its lowest possible value (deviating from that alignment causes a restoring force which pushed the spin back into alignment, so this state has low potential energy), while when anti-aligned the total energy is maximized. This is exactly represented by this two-state model.

At low temperature, all spins will be lined up with the magnetic field and are in their lowest energy state. This makes sense because at low/nearly zero temperature there is no thermal motion that can shake the spin into the anti-aligned state. As temperature increases, the spins get more randomly oriented (half aligned, half anti-aligned) due to thermal motion.


Problem *SS-10* (25 Points)

Consider the following potential well, containing the energy levels indicated ($n = 1, 2, 3$ and $E = E_0, 2E_0, 3E_0$). You have 5 electrons that you can place in this well.
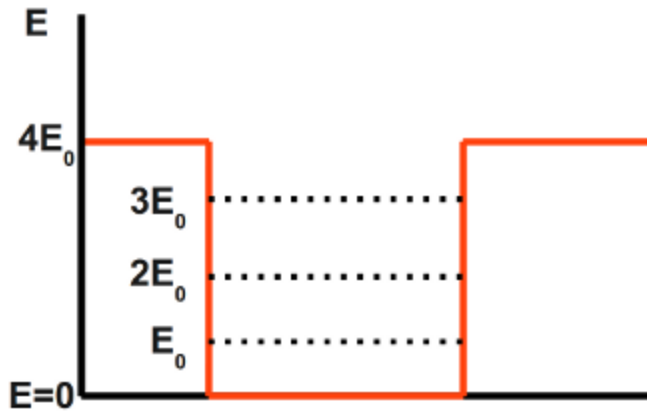
1. Make your own drawing of the potential well, showing (e.g. by sketching circles on the energy levels) where the 5 electrons will be when the system has the lowest possible total energy. Next to each electron circle in your drawing, label the state of that electron using all of its quantum numbers.

   SOLUTION
   The top-most electron can be labeled by either $(3, \pm 1/2)$.
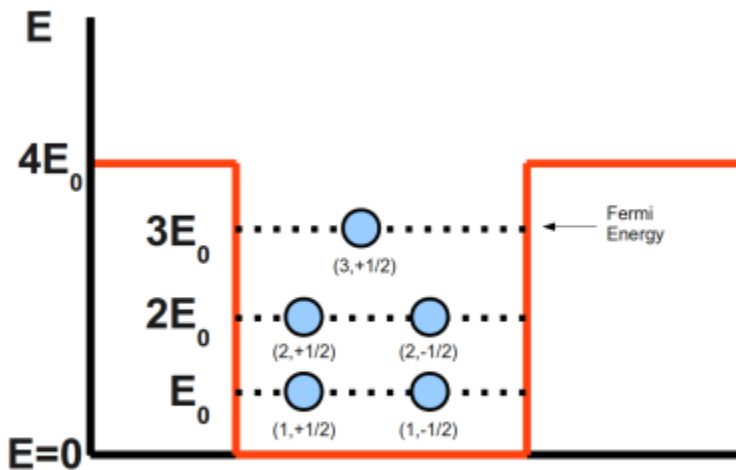
2. What are the occupation numbers corresponding to $E = E_0$, $E = 2E_0$, and $E = 3E_0$?

SOLUTION
The occupation number is the number of particles per state. Since these are fermions, and no more than 1 fermion can be in a state, the occupation numbers are all 1 except (3,-1/2), which has zero occupation number. At each energy level, there are 2 states available.

3. Indicate on your drawing where the Fermi energy is located.

SOLUTION
See the above drawing. This occurs where the occupation number falls to 1/2. Since in this case there is one filled state of two possible states in $3E_0$, it is here that the occupation number drops to 0 and this energy corresponds to the Fermi energy. In general, the Fermi energy always corresponds to the highest-energy state with at least 1 fermion in the energy level.

4. What is the work function of the system described by this well?

SOLUTION
The work function is the energy required to remove the top-most electron from the well. This energy is $\phi = 4E_0 - 3E_0 = E_0$.

HARRIS *CH9-66* (30 Points)

Part a: the Fermi velocity is given in the relationship $E_F = \frac{1}{2}m\nu_F^2$. The Fermi energy for conduction electrons in sodium is 3.1 eV. Thus

$$\nu_F = \sqrt{2E_F/m} = 1.0 \times 10^6 \text{ m/s.}$$

Part b: The wavelength of such an electron is given by
$\lambda = h/p = h/m\nu_F = 7.0 \times 10^{-10}$ m.

Part c: If each sodium atom contributes one conduction electron and sodium atoms are spaced 0.37 nm apart in the solid, is it necessary to treat these electrons as a quantum gas? The wavelength in nm is 0.70 nm, just a factor of 2 larger than the atomic spacing. It is therefore necessary to treat this as a quantum gas, since the relevant dimensions of the atomic spacing are comparable to the wavelength of the conduction electrons ($\lambda << d$ does NOT hold).