Modern Physics (PHY 3305) Lecture Notes

HomeworkSolutions009

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Point Distributions

no tags

Points were distributed as follows for each problem:

Problem	Total	Point Distribution
CH11-13	10	Recognized that you could compute the volume of an atom from the density (4 Points), computed the volume of the nucleus (4 Points), and took the ratio to solve the problem (2 Points)
CH11-14	10	Used energy conservation (5 Points) and computed the kinetic and potential energies (2 Points), using the balancing of the two at the nuclear radius (2 Point) to solve the problem. Tested for relativistic effects, given the high speed (1 Point)
CH11-21	20	Explained a means by which they estimated the number of spheres maximally in contact with a single sphere (15 Points), correctly obtaining 12 (2 Points), and then solving for the BE/nucleon (3 Points)
CH11-23	5	Used the relationship between BE and hydrogen/nucleon/atomic masses to solve for the BE, or employed the semi-empirical formula (5 Points)
SS-12	5	Used the semi-empirical formula to solve for the BE (3 Points) and converted that to mass (2 Points)
CH11-46	40	Used the semi-empirical formula to compute the BE and/or the mass of the parent nucleus (25 Points), and then used that same approach to either compare binding energies or compute the Q-value for the proposed reactions (5 Points each)
CH11-54	10	Part (a): used the relationship between decay rate and the decay constant to compute the number of nuclei (5 Points)

Part (b): used the radioactive decay law to compute the age of the sample (5 Points)

Deductions outside of the above:

- 1 point is deducted for failing to box the numerical answer
- 1 point is deducted for incorrectly applying significant figures
- other points are deducted as outlined in the homework policy

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HARRIS, *CH11-13*

SOLUTION

Solving this problem boils down to comparing the volume of the nucleus of iron to the size of an iron atom, which can be determined from the density of iron. The nuclear volume can be written as $4/3\pi r^3$, where $r = \sqrt[3]{A}R_0$. Iron's mass number is 56, so the radius is $4.0 \times 10^{-43}\ \mathrm{m}^3$. The volume for each atom per nucleus is given by taking the atomic mass of iron and dividing by the density of iron, yielding the volume per atom. The atomic mass is 55.847u, and there are $1.66 \times 10^{-27}\ \mathrm{kg/u}$. Thus the volume per atom is $1.2 \times 10^{-29}\ \mathrm{m}^3$. The ratio of the volume of the nucleus to the volume of the atom is then 3×10^{-14} - indeed, most of the mass of the atom is concentrated into just a tiny fraction of its overall volume!

HARRIS, *CH11-14*

SOLUTION

The key to solving this problem is energy - the kinetic energy of an alpha particle pitted against the Coulomb potential energy of the repulsion due to the nucleus. The kinetic energy of an alpha particle, for now, will be assumed to be classical (we will check this, however!): $KE = \frac{1}{2}mu^2$. We want to solve for the u required to get the alpha particle to just touch the nucleus, at which point the attractive strong nuclear interaction can take over. The potential energy of the alpha particle in the electric field of the nucleus will be $-kq_\alpha(Z_{Au}e)/r$, where $r = \sqrt[3]{A}R_0,\ A = 197,\ Z = 79$, and $q_\alpha = +2e$. The alpha particle will JUST contact the radius of the nucleus when it has kinetic energy equal to the potential energy. Therefore, we only need to solve for u in:

$$\frac{1}{2}mu^2 = kq_\alpha(Z_{Au}e)/r.$$

Solving for u^2:

$$u^2 = 2 \times 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \times \frac{2 \times 79 \times (1.6 \times 10^{-19} \text{ C})^2}{(4 \times 1.66 \times 10^{-27} \text{ kg})(\sqrt[3]{197} \times 1.2 \times 10^{-15} \text{ m})}$$

This yields:

$$u = 4.0 \times 10^7 \text{ m/s}$$

Ah . . . this is about 13% the speed of light! We should be careful, and try using the fully relativistic kinetic energy to check the answer:

$$(\gamma_u - 1)mc^2 = kq_\alpha(Z_{Au}e)/r$$

$$\gamma_u = 1 + 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \times \frac{2 \times 79 \times (1.6 \times 10^{-19} \text{ C})^2}{(4 \times 1.66 \times 10^{-27} \text{ kg})(\sqrt[3]{197} \times 1.2 \times 10^{-15} \text{ m})(2.998 \times 10^8 \text{ m/s})^2}$$

which yields

$$u = 0.131c = 3.9 \times 10^7 \text{ m/s}$$

and so while the classical approach was "good enough," you see that even in nuclear physics you need to be aware of your relativistic surroundings.

HARRIS, *CH11-21*

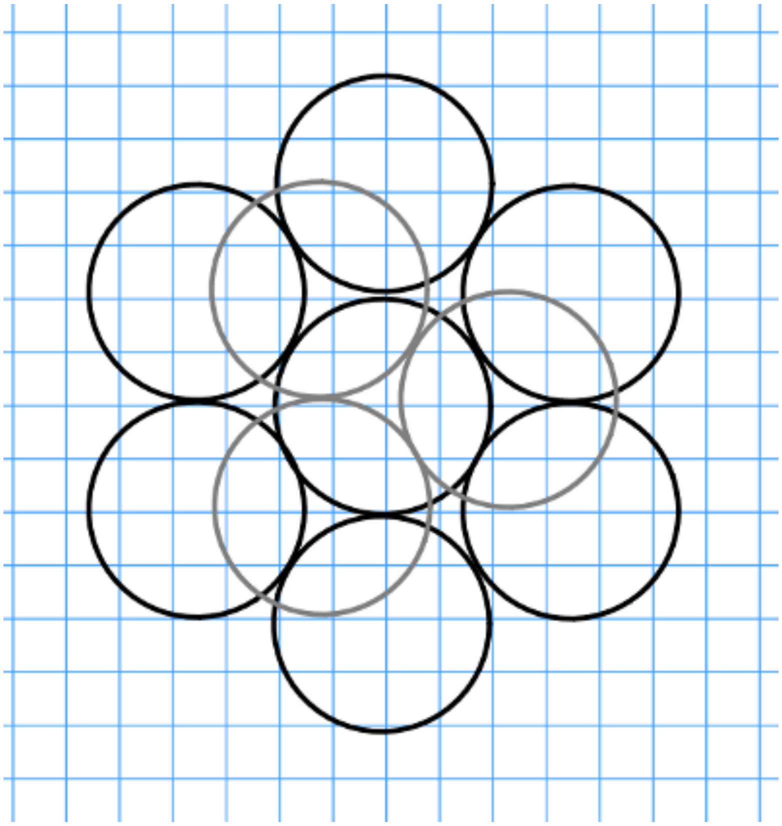
SOLUTION

There are many ways to solve this problem which are acceptable to an empirical scientist such as your instructor. One way would be to start in 2-D and think about circles. For instance, take a bunch of same-denomination coins from your pocket and lay them on a piece of paper. See how many you can arrange around a central coin, maximally, so that all of the surrounding coins make contact with the central one. You should find the answer to be 6, in 2-D (we'll get to 3-D in a moment).

Another approach, again starting in 2-D, would be to think about what "touching" means for geometrical objects of equal radius. Since the distance from the central circle to the center of a circle in contact with it will be 2R (twice the radius of either circle). This is the MINIMUM distance between the centers of any two circles surrounding the central circle; now you just have to figure out how many times you can place circles of radius R, each touching the central circle (so that they are 2R from each others' centers), around the central circle. The answer is 6 - a hexagonal pattern of 2R arranged around the central circle, with each "corner" of the hexagon lying 2R from the center of the central circle.

Now, think in 3-D. Imagine that your 2-D arrangement is a plane, sandwiched between two other planes also containing spheres (instead of circles). How many spheres in the plane above the original one can be placed in contact with the original sphere? The answer is 3 (see drawing below), arranged such that they form a triangle in 2-D in that plane with sides of length 2R. The plane below the middle plane will likewise contain 3 sphere touching the original sphere, for a total of 12.

Since each nucleon gets 1/2 a bond, that makes for 6 bonds maximum that a surrounded nucleon can have. If we consider the deuteron as representative of any bonded pair of nucleons, and each nucleon in a deuteron has 1/2 a bond, then this is 12 times the bonds in the deuteron, and thus 12 times the binding energy per nucleon in a deuteron. Since the binding energy per nucleon is about 1 MeV, that means that the maximum BE/nucleon would be 12 MeV for our packed sphere model. How does that compare to Fig. 11.4? That figure shows a maximum BE/nucleon of 9 MeV, less than our model allows. This suggests that our model is incomplete - for instance, it misses the repulsive Coulomb force which grows as the number of nucleons grows in stable nuclei, and which serves to REDUCE the BE/nucleon. That might explain why we get an overestimate.



HARRIS, *CH11-23*

To calculate the BE/nucleon of Carbon-12 (A = 12, Z = 6), we already have the measured atomic mass, so we can employ the "simple" formula:

$$BE = (Zm_H + Nm_n - M_{^A_Z X})c^2$$

where $m_H = 1.007825u$, $m_n = 1.008665u$, and $M_{^{12}_6 C} = 12u$. Thus:

$$BE = 92.2MeV$$

Since Carbon as 12 nucleons,

$$BE/A = 7.68 \text{ MeV/nucleon}$$

SS-12

SOLUTION

Instead of using the measured mass of Carbon-12, let's pretend we know nothing about Carbon-12 except its mass number, Z, and N. We can then use the semi-empirical formula to get the binding energy, and from the binding energy we can compute the mass of Carbon-12.

The semi-empirical formula states that:

$$BE = 15.8 \times A - 17.8 \times A^{2/3} - 0.71 \times \frac{Z(Z-1)}{A^{1/3}} - 23.7 \times \frac{(N-Z)^2}{A}$$

Plugging in our A, N, and Z for carbon (A=12, Z = 6, N = 6):

$$BE = 86.998 MeV$$

From this, we can compute the hypothetical mass of a Carbon-12 nucleus:

$$M_{^{12}_{6}C} = Zm_H + Nm_n - BE/c^2 = (6 \times 1.007825u + 6 \times 1.008665u - 0.093396u = 12.006u$$

(it's very convenient to remember that $c^2 = 931.5 \text{ MeV/u}$)

This predicted mass compares extremely well with the known mass of $12u$ for Carbon-12.

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HARRIS, *CH11-46*

SOLUTION

You are given a hypothetical nucleus, $^{288}_{119}X$, and asked to determine whether energy would likely be released under different decay hypotheses. Let's begin by computing the binding energy of this nucleus, which we'll likely need going forward.

$$BE = 15.8 \times A - 17.8 \times A^{2/3} - 0.71 \times \frac{Z(Z-1)}{A^{1/3}} - 23.7 \times \frac{(N-Z)^2}{A}$$

$$BE = 15.8 \times 288 - 17.8 \times (288)^{2/3} - 0.71 \times \frac{119 \cdot 118}{(288)^{1/3}} - 23.7 \times \frac{(169-118)^2}{288} = 2058.7 \text{ MeV}$$

The mass of this nucleus can then be determined:

$$M_{^{288}_{119}X} = Zm_H + Nm_n - BE/c^2 = 288.19u$$

We know from the principle of energy minimization that any final state that contains more total energy than this binding energy will be "unfavorable," and thus not likely to occur. Let's test each decay mode proposed in the problem.

Part (a): Alpha Decay

Can this nucleus alpha decay? The reaction would be:

$$^{288}_{119}X \rightarrow ^{284}_{117}Y + ^{4}_{2}\alpha$$

We'll need the mass of the $^{284}_{117}Y$ nucleus, computed similarly to the above mass of the X nucleus (from the semi-empirical binding energy formula):

$$BE_Y = 2043.5 \text{ MeV}$$

$$M_{^{284}_{117}Y} = 284.17u$$

The Q-value for this decay is:

$$Q = (m_i - m_f)c^2 = (288.19u - (284.17u + 4.002603u)) \times 931.5 \text{ MeV/u} = 16 \text{ MeV}$$

Q is positive, so this reaction is exothermic (there is kinetic energy left over after the decay), and so we would expect this isotope to decay.

An alternative way to arrive at the answer is simple to compare the total binding energy of the initial and final state. For that, we need the binding energy of the alpha particle, which is:

$$BE_\alpha = 28.3 \text{ MeV}$$

Comparing the initial and final state binding energies:

$$BE_f - BE_i = (2059 - (2044 + 28)) = 13 \text{ MeV}.$$

The BE of the final state is larger than that of the initial, meaning it's a more tightly bound system and thus at a lower overall energy than the initial state.

Part (b): β^+ decay

The decay equation is:

$${}^{288}_{119}X \rightarrow {}^{288}_{118}Y + 0_{+1}\beta^+ + \nu$$

The neutrino is, for our purposes, a zero-mass object. Therefore, it doesn't contribute to the mass of the final state. We need only compute the mass of our new isotope, Y, and the mass of the positron, and add them together.

$$BE_{{}^{288}_{118}Y} = 2067.3 \text{ MeV}$$

$$M_{{}^{288}_{118}Y} = 288.18u$$

The Q-value for this reaction is:

$$Q = (m_i - m_f)c^2 = (288.19u - (288.18u + 5.5 \times 10^{-4}u + 0u)) \times 931.5 \text{ MeV/u} = 9.3 \text{ MeV}$$

Which is small, but still positive. This reaction is, indeed, possible.

Part (c): β^- decay

The decay equation is:

$$^{288}_{119}X \rightarrow ^{288}_{120}Y + 0_{-1}\beta^{+} + \nu$$

We compute the mass of our new isotope, Y, and the mass of the positron, and add them together.

$$BE_{^{288}_{118}Y} = 2049.2\ \mathrm{MeV}$$

$$M_{^{288}_{118}Y} = 288.19u$$

The Q-value for this reaction is:

$$Q = (m_i - m_f)c^2 = (288.19u - (288.19u + 5.5 \times 10^{-4}u + 0u)) \times 931.5\ \mathrm{MeV/u} = -0.51\ \mathrm{MeV}$$

Which is small, but NEGATIVE. This reaction is not possible - the cost in gaining a proton outweighs any other energy benefit.

HARRIS, *CH11-54*

SOLUTION

Part (a): if a specimen has a carbon-14 decay rate of 3.0 Hz ($3.0\ \mathrm{s}^{-1}$), then we can use the radioactive decay formula to determine the number of carbon-14 nuclei present in the sample. The decay constant for carbon-14, given in the example on page 501-502, is $\lambda = 3.83 \times 10^{-12}\ \mathrm{s}^{-1}$. The number of nuclei is given by

$$N = \frac{R}{\lambda} = \frac{3.0\ \mathrm{s}^{-1}}{3.83 \times 10^{-12}\ \mathrm{s}^{-1}} = 7.83 \times 10^{11}$$

Part (b): If this represents 1/10 of the number of nuclei present when the sample originally died, we can date the sample using the radioactive decay law:

$$N = (1/10)N_0 = N_0 e^{-\lambda t}$$

$$-ln(0.1)/(3.83 \times 10^{-12}\ \mathrm{s}^{-1}) = t = 6.01 \times 10^{11}\ \mathrm{s} = 19,000\ \mathrm{years}$$