Modern Physics (PHY 3305) Lecture Notes

Relativistic Paradoxes and Kinematics (Ch. 2.4-2.5)

SteveSekula, 26 January 2010 (created 13 December 2009)

tags: lecture

CHAPTER 2.4-2.6

Review of last class

- We learned the postulates of Einstein, made to resolve the apparent discrepancy between electromagnetism and classical mechanics
- We learned about the consequences of the postulates
- We learned how to relate events in one frame to those in another (the Lorentz Transformation)
- We applied the Lorentz transformation, and explained one observed phenomenon: muons reach the earth

Today's Material

- Rigorous Applications:
  - The Life of the Muon
  - Paradoxes in Relativity (there are none)
- Motion in Relativity
  - the effect of motion on light
  - transformation of velocities in special relativity

Putting it all together: the life of the muon

We can now rigorously apply what we have learned to gain a deeper understanding about the observation that muons, which only live $2.2\mu\text{s}$, are able to reach the surface of the earth.

Let's make this concrete.

What is a muon?

- It is a particle that is not part of atoms. It was first discovered in 1936 by Carl Anderson. He saw them as part of something called "cosmic ray showers," caused when high energy particles from outer space (cosmic rays) strike the atmosphere and rain down showers of particles on the earth.

- Today, we believe that the muon is one of 12 fundamental subatomic particles that represent the building blocks of the universe.
- As far as we know, there is nothing "inside" the muon; that is, it is *fundamental* and not made of other things
- It lives for a grand total of 2.2μs once it is created, and when it decays ("dies") it emits an electron and other fundamental subatomic particles called "neutrinos"
- There is an experiment on display in the basement of Fondren Science that counts muons and measures their lifetime.

Assume a muon is created 3.0km above the surface of the earth (the atmosphere is about 37km thick). It's traveling at 0.98c when it is created, nearly the speed of light! Let's think about frames of reference first.

1. Let's define frame S as the surface of the earth, and the positive x direction along the direction of flight of the muon. That means the muon moves at speed $+\nu$. We are standing on the earth and observing this muon as it travels from where it is created toward us on the earth.
2. Let's define the frame of the muon as S', again with the positive x' direction in front of the muon so that the speed of the earth in that frame is $-\nu$. What does the muon "see", if it could see anything? The muon sees this planet suddenly appear in front of it, racing toward it at very high speed! Of course, it argues it's standing still and the earth is racing toward it and it's perfectly legitimate for it to say this. We'll take a specific perspective about whom is moving and whom is at rest.

Let's now ask and answer some questions about the muon in this problem.

1. Let's take a *classical physics view* for just a moment. In classical physics, time is the same for the muon and for the observer on the earth. Let's ask and answer the following question from this perspective: How far does the muon travel before it decays?
   a. Let's begin by thinking about the distances in classical physics. To measure a distance, I will simultaneously locate the start and finish points. In classical physics, since time is the same for all frames of reference, that means that what is simultaneous in frame S is also simultaneous in frame S'. In math, we express such simultaneity as $dt = dt' = 0$ - that is, when I measure length in any frame I make that measurement all at once. DISTANCE is defined as the difference in spatial coordinates between one place (x_1) and another (x_2), or

$L = dx = x_2 - x_1$. Applying calculus to the Galilean Transformation,

$$x' = x - \nu t$$

we find that

$$dx' = \left(\frac{dx'}{dx}\right) dx - \left(\frac{dx'}{dt}\right) dt = dx - \nu dt.$$

Since $dt = dt' = 0$ for our length measurement, in either frame, we find that

$$dx = dx'$$

and measurements of length in one frame are the same as any other in classical physics.

b. That means that classically the muon and the observer agree that the distance to the earth is 3.0km, regardless of their state of motion. In fact, they'll agree that any distance in one frame is the same in the other. The answer to the problem above can now be found.

c. Since time is assumed to be the same for both the muon and the observer on earth, and distances are thus the same for both, we can do the problem in one frame and know the answer in the other with no additional work. The person on earth knows from experiments with muons that the muon lifetime is 2.2μs. So how far does it travel in 2.2μs at $0.98c$? Well, we know that speed, distance, and time are all related as:

$$u = \frac{dx}{dt} = L/\Delta t.$$

We thus plug in our numbers and solve for distance:

$$L = u \cdot \Delta t = (0.98 \times 2.995 \times 10^8 m/s) \times (2.2 \times 10^{-6} s) = 650m.$$

d. Does the muon make it to the surface of the earth? **Classically: NO.** In classical physics, where space is relative but time is absolute, and distances are the same for all observers, the muon could never even get close to making it 3km to the surface of the earth. It doesn't even get 1/3 of the way! And yet, we know that copious numbers of muons make it to the surface of the earth.

e. How much longer would the muon have to live to make it the full 3km? Well, we know that in 2.2μs the muon goes 650m, so for how long would it have to live to go 3km? We apply our formula again:

$$\Delta t = L/u = (3.0 \times 10^3 m)/(0.98 \times 2.995 \times 10^8 m/s) = 10.\mu\text{s},$$

or about 8μs longer than its nominal lifetime.

1. Let's now apply Special Relativity to the problem and see what happens.
   a. Let's first think about what special relativity does to the story of the muon from the perspectives of frame S and frame S'. In frame S, our earth observer, the story is as follows: "The muon is traveling at a very high speed, close to light. Therefore, for it, time is passing more slowly, which means its clocks (if it could have clocks) would be running slowly. Therefore, although in its frame of reference 2.2μs passes before it decays, in my frame I see the muon's clock running slowly and to me it lives longer. Thus, it will travel further than 650m." The story in frame S' is a little different: "The muon suddenly comes into being and sees this planet in front of it. The planet is moving toward it at a great speed, close to that of light. Because of that, distances are shorter than they are in frame S and in its 2.2μs of being in existence it covers that distance easily." So while the story is different in the two frames, the events are not in question: the muon appears, goes far, and then disappears.
   b. Let's explore this numerically using special relativity. Again, in frame S we have the equations

$$x' = \frac{1}{\sqrt{1-(\nu/c)^2}}(x - \nu t) \;\text{ and }\; t' = \frac{1}{\sqrt{1-(\nu/c)^2}}(-\frac{\nu}{c^2}x + t)$$

and in frame S' we have the equations:

$$x = \frac{1}{\sqrt{1-(\nu/c)^2}}(x' + \nu t') \;\text{ and }\; t = \frac{1}{\sqrt{1-(\nu/c)^2}}(\frac{\nu}{c^2}x + t).$$

The gamma factor is defined as

$$\frac{1}{\sqrt{1-(\nu/c)^2}} = 1/\sqrt{1-\nu/c^2} = 1/\sqrt{1-0.98^2} = 5.0.$$

      i. From the perspective of frame S, how long does the muon live? Well, to figure that out we apply the Lorentz Equations in frame S:

$$\Delta t = \frac{1}{\sqrt{1-(\nu/c)^2}}(-nu/c^2 \cdot \Delta x' + \Delta t')$$

and in particular, we apply the fact that the muon in its frame is standing still and the cosmos is moving around it, so $dx' = 0$ for the muon. Thus,

$$\Delta t = \frac{1}{\sqrt{1-(\nu/c)^2}} \Delta t' = 5.0 \times 2.2 \times 10^{-6} s = 11 \mu\text{s}.$$

That is, from our perspective on the earth, the lifetime of the muon at $0.98c$ appears to be $11\mu\text{s}$, not $2.2\mu\text{s}$.

ii. From the perspective of frame S, the muon lives a lot longer. Since classically it had to live $10\mu\text{s}$ to reach the surface of the earth, the fact that it lives $11\mu\text{s}$ means it not only makes it to the earth - it passes into it and keeps going! How far does it go? That one's easier:

$$\Delta x = \Delta t \times u = 11\mu\text{s} \times 0.98 \times 2.995 \times 10^8 m/s = 3.2 km,$$

so it goes 200m into the earth before it decays.

iii. What about the perspective from frame S'? Well, we again need to apply the Lorentz Transformation in that frame to figure out what the muon "sees." The equation of interest is

$$\Delta x = \frac{1}{\sqrt{1-(\nu/c)^2}} (\Delta x' + \nu \Delta t').$$

The muon sees itself as standing still and the cosmos is moving around it, so for it $dx' = 0$. Thus,

$$\Delta x = \frac{1}{\sqrt{1-(\nu/c)^2}} \cdot \nu \cdot \Delta t'.$$

We can now solve the time that the muon measures before it reaches the surface of the earth,

$$\Delta t' = \Delta x / (\nu \cdot \frac{1}{\sqrt{1-(\nu/c)^2}}) = 3.0 \times 10^3 km / (0.98 \cdot 2.995 \times 10^8 m/s \times 5.0) = 2.0 \mu\text{s}.$$

So for the muon in its frame, only $2.0\mu\text{s}$ are required to reach the surface of the earth, which means it penetrates into the earth and decays $0.2\mu\text{s}$ later.

The bottom line is that the application of the Lorentz Transformation in either frame is in agreement with Einstein's postulates. Both the muon and the observer on earth agree that the muon enters the earth before decaying. They may disagree on how that's happening, but it happens.

The Twins Paradox

The Twins Paradox was an early objection to relativity, and can be stated as such:

If one of two twins stays on Earth while the other travels away at high speed. Each argues that the other is aging more slowly. When the traveling twin returns, each will argue that the other much be the younger twin.

Thus the paradox: it pains the mind to conceive of this possibility, yet it naively seems a consequence of special relativity..

A crucial factor is overlooked: the traveling twin will have to RETURN home at some point, turning around and coming back. That means changing from one inertial reference frame to another, and heading back to earth. Her observations won't be the reliable ones. Let's explore this.

Consider a pair of fraternal twins, Anna and Bob, born in the future. An incident on earth necessitates their separation at birth; Bob remains behind with his mother, and Anna and her father are put on a spaceship headed for the safety of Planet X, 40 light-years away. Their ship travels at 0.8c. They remain in contact with their earth-bound family members via light signals. The annual birthday messages from earth are very popular amongst the refugees on the ship.

1. What is the definition of our frames?
   a. To find the root of the paradox, we have to assume that each person is justified in arguing they are at rest. We'll see then why people think there is a paradox.
2. What is the gamma factor involved here?
   a. $\frac{1}{\sqrt{1-(\nu/c)^2}} = \sqrt{1/(1-\nu^2/c^2)} = 1.67$
3. In Bob's frame, how old is he when Anna reaches Planet X?
   a. Bob knows the cruising speed of the ship is 0.8c, and that Planet X is 40 ly away. Thus he is

   $$\Delta t = L/\nu = 40\text{yr}\cdot c/(0.8c) = 50\,\text{yr}$$

   old when the ship arrives at Planet X.
4. How old will Anna be when the ship reaches Planet X?
   a. Bob argues that since Anna is moving and he is at rest, her clocks run more slowly and thus the Lorentz Transformation equation that needs to be applied to get her age is

   $$\Delta t' = \frac{1}{\sqrt{1-(\nu/c)^2}}(-\nu/c^2\Delta x + \Delta t).$$

Since for him, the distance she travels is $\Delta x = L = 50\text{yr} \cdot c$, then he calculates that Anna will be

$$\Delta t' = 1.67 \times \left(-0.8c/c^2 \cdot 40\text{yr} \cdot c + 50\text{yr}\right) = 30\text{yr}.$$

An alternative way to argue this from Bob's perspective is to state that since Anna is moving and the events (leaving earth and arriving at Planet X) happen at the same place for her her time must be the *proper time*. Thus

$$\Delta t / \frac{1}{\sqrt{1-(\nu/c)^2}} = \Delta t_0 = 30\text{yr}.$$

Same answer, different reasoning (same physics!).

5. How old will Anna be when the ship reaches Planet X?
   a. In Anna's frame, where she is at rest and sees the earth pass by, then later planet X, at with velocity $-\nu$, the correct Lorentz Transformation equation to apply to the problem is

$$\Delta t' = \frac{1}{\sqrt{1-(\nu/c)^2}}(\nu/c^2 \Delta x + \Delta t).$$

. For her, $\Delta x' = 0$. Thus we can solve for $\Delta t'$:

$$\Delta t' = \Delta t/(\frac{1}{\sqrt{1-(\nu/c)^2}}) = (40\text{yr} \cdot c/0.8)/1.67 = 30\text{yr}$$

6. How old does Anna say Bob is when she reaches Planet X?

a. To Anna, Bob is in motion at velocity $-\nu$. The correct Lorentz Transformation equation here is $\Delta t = \frac{1}{\sqrt{1-(\nu/c)^2}}(\nu/c^2 \Delta x' + \Delta t')$. Since in his frame, Bob is not moving anywhere $\Delta x' = 0$. Thus

$$\Delta t' = \frac{1}{\sqrt{1-(\nu/c)^2}} \Delta t$$

and

$$\Delta t = \Delta t' / \frac{1}{\sqrt{1-(\nu/c)^2}}.$$

. Thus Anna determines Bob's age at the time of her arrival at Planet X to be

$$\Delta t = 30\,\mathrm{yr}/1.67 = 18\,\mathrm{yr}$$

. This is truth from her perspective!

7. Let's explore one more thing. What if, 10 years after the incident, the crisis abates? Bob and his mother send a signal out to Anna's ship, sending greetings (and saying that Bob is 10 years old) and asking them to return to Earth as soon as they get the message.
   a. How long does that message take to reach Planet X?
      i. From Bob's frame ,the message travels at c and takes 40 years to reach Planet X
      ii. From Anna's perspective, Planet X is approaching the light signal from earth (which travels at c in her frame, too) at a speed of 1.8c; that is, the distance between them decreases at a rate of 1.8c. Since the distance from Earth to planet X is 24 ly, it takes 13 years for the signal to make the trip. As a side note, Anna calculates that since Bob said he was 10 when the signal was sent, and since she argues Bob's clocks are running slow, he would have only aged $13\mathrm{yr}/1.67 = 8\mathrm{yr}$ in the time since the signal was sent, making him 18 years old now. That's consistent with our earlier calculation

But we can't stop here! This is where people usually stop and declare a paradox has happened. There would only be a paradox right now if Bob and Anna could be in instantaneous contact right now, or if they were face-to-face - but neither of those are true (or possible), so we have to finish the story. The only way to do this problem is all the way through. Let's look at the story from each person's perspective, in tum:

- **Bob**
  - Bob is 0 years old when Anna and he are separated. She leaves on a ship at 0.8c. When Bob is 10 years old, the crisis on earth abates and he and his mother send a message to Planet X, in the hopes that it will reach Anna and their father by the time they arrive. At c, the message take 40 years to year planet X, so just as Anna is supposed to arrive there they know the message should be received. Bob assumes that Anna then turns right around and comes back. He knows that he won't see her for 50 more years. He is 100 when her ship lands on earth, and she appears to him to only be 60 years old. Talking to her, he indeed confirms that the first leg of the journey seems to take only 30 years. Carla, the captain of the vessel that brought Anna back, confirms that her trip from Planet X to earth took 30 years (we'll discuss why we want that information from Carla and not Anna in a moment). Bob is 100, Anna is 60. That's the story from Bob's perspective, and the events (Bob's age and Anna'a age) are not in dispute.

- **Anna**
  - Anna is 0 years old when she and her father flee earth for Planet X. When they arrive at Planet X, they receive a message from Bob and their mother. In the message, Bob is 10 years old, but Anna's father is clever and knows Bob's not 10 now. The message had to travel to them from earth at c, and Planet X was approaching at $0.8c$, making the light and Planet X approach each other at 1.8c from her perspective. They immediately ask to be transferred at velocity to a merchant marine vessel headed to earth and captained by Carla. Carla has been traveling at 0.8c toward earth, and has been inertial the whole time. During the transfer, Anna and her father must accelerate and since we don't know how acceleration affects perceptions we cannot trust their observations anymore. Carla can be trusted, since she's been inertial this whole time. She informs them that based on her calculations, Bob must be 82 years old (we'll get to that in a second). Since the return trip takes 30 years, during which time Bob ages another 18 years, Anna concludes that Bob will be 100 years old when she arrives.

- **Carla**
  - Carla has been traveling back toward Earth on a supply run. As she approaches Planet X, she is asked to take on passengers at velocity. She also receives that transmission from Bob and his father, in which

Bob is 10 years old. Carla also measures the distance from earth to Planet X to be 24 ly, and since the signal was moving toward Carla at c and Planet X was moving toward Carla at 0.8c, she concludes light approached Planet X at $c - 0.8c = 0.2c$. She concludes the transmission required $t = 24ly/0.2c = 120\mathrm{yr}$ to travel the distance. Carla argues Bob's clocks are running slower, so she concludes he has aged $t = 120y/1.67 = 72y$ since he sent the transmission, making his total age 82 years. She tells Anna and her father this when they arrive on board. Knowing that Bob will age another 18 years during the 30 year return trip to Earth, they conclude that Bob will be 100 years old when they arrive.

Applying ALL of the consequences of Einstein's postulates is critical when thinking about relativity. In short:

- In special relativity, which ONLY deals with inertial reference frames, you cannot assume acceleration has no effect on observation and thus you should ONLY trust computations performed in inertial frames of reference
- While no single velocity can exceed that of light, and light always travels at c for all observers, RELATIVE velocities can exceed that of light. This violates nothing about causality, because all that matters is that relative to the OBSERVER objects cannot move faster than c.
- Simultaneity, length, and time are all RELATIVE concepts. You cannot assume that any of these holds for all observers.

Kinematics

We now investigate the further consequences of special relativity. We'll begin by revisiting the Doppler Effect, for light instead of sound. It's a critical consequence of both classical and modern physics. We'll then dive into the transformation of velocities, which we last discussed before deriving the Lorentz transformation. We'll then discuss the consequences of the transformation of velocity on energy and matter, and come to one of the most stunning predictions of special relativity.

The Doppler Effect

The Doppler Effect is known to the student of classical physics through its implications for sound. In classical physics, sound experiences a pitch shift (frequency shift) when the sound source is moving relative to the observer (listener). When the sound source moves away from the listener, the time between amplitude peaks/troughs is made longer by the motion of the source; when the sound source moves toward the observer, the time between amplitude peaks/troughs is shorter and thus the frequency increases. Imagine the simple case of a sound source moving along a straight line toward or away from a person (speed is $\pm\nu$). The Doppler Shift in frequency is given as follows:

$$\Delta t = \Delta t' + \frac{\nu}{c_{sound}}\Delta t' \rightarrow f \equiv 1/\Delta t = \frac{f'}{(1+\frac{\nu}{c_{sound}})}$$

If the source is moving at an angle to the listener, rather than just toward or away, then:

$$f = \frac{f'}{(1+\frac{\nu}{c_{sound}}\cos\theta)}$$

Thinking classically for a moment longer, does the same happen for light?

The answer is yes! Even ignoring Einstein and his colleagues for a second, there is a Doppler Effect for light in classical physics. Light is emitted by oscillating electric charges (at some frequency f' in the rest frame of the source), and if such a source is moving away from you you'll measure a frequency given by:

$$f = \frac{f'}{(1+\frac{\nu}{c}\cos\theta)}$$

Often, the Doppler Effect is introduced in the context of relativity as if, for light, it only happens in special relativity. Classical physics admits this perfectly well. However, relativity adds a new twist.

If the source is moving, time passes more slowly for it, which means oscillating charges do so more slowly. This adds an additional effect to the frequency:

$$\Delta t' \rightarrow \gamma_\nu \Delta t' \rightarrow f'/\gamma_\nu$$

This gives the full form of the Doppler Effect as:

$$f = f' \frac{\sqrt{1 - \frac{\nu^2}{c^2}}}{(1 + \frac{\nu}{c}\cos\theta)}$$

Let's think about what this means:

- In classical physics, moving away from the source "red-shifts" (lowers the frequency) of the light while moving toward the light "blue-shifts" it.
- In modern physics, the above is true but NO MATTER WHAT, there is an inherent red-shift in the light regardless whether the source is moving toward or away from the observer. This is because time moves more slowly for the moving object regardless of its direction of motions, and the slowed movement of the charges lowers the light frequency.

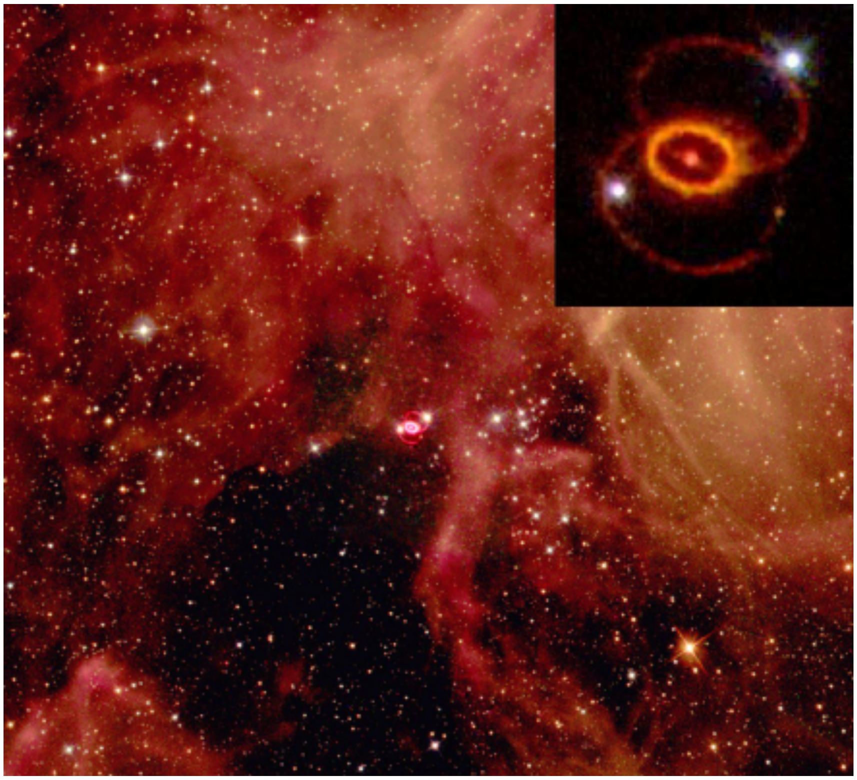
Let's compare the relative effects for two cases:

1. You are driving toward a traffic light at 0.15c. A policeman next to the light sees it as red (650nm). What color does the light appear to you?

    - 650nm light has a frequency given by $f = c/\lambda = 4.6 \times 10^{14}\mathrm{Hz}$. We can plug this into the formula to find that the frequency observed by you, the driver, is $5.4 \times 10^{14}\mathrm{Hz}$, which corresponds to a wavelength of about 560nm. This is green light, so you blow through the traffic light thinking it's green! The cop nabs you, and you both argue whether relativity has any place in modern law.

We see that even at "modest" speeds, (0.15c), a dramatic shift in the character of light can occur (on the scale of the human eye). Is this speed common in our world? 0.15c is about 10 million mph. Does that speed occur in nature?

ANSWER: yes, very common. Once per decade per galaxy, a star explodes in a cataclysm called a "supernova". The hydogen gas blown off the surface of the star expands outward at about 10 million mph (33 million mph for *SN1987a*). Supernovas are factories for the heavy elements in the universe. Without these common phenomena, it would not be possible to produce worlds like ours out of gravitational collection of interstellar gas.



Next Time:

- Transformation of velocities - seems "bread and butter", but it has deep implications for cherished principles in physics: energy and momentum conservation
- We'll learn about the nature of energy and mass
- We'll discuss the wave nature of light and prepare for understanding a parallel revolution in physics: quantum physics.