Modern Physics (PHY 3305) Lecture Notes

Velocity, Energy and Matter (Ch. 2.6-2.7)

SteveSekula, 29 January 2010 (created 13 December 2009)

tags: lecture

CHAPTERS 2.6-2.7

Review of last lecture

- We explored one of the popular so-called "paradoxes" of special relativity and realized that there is no paradox when Einstein's postulates are applied thoroughly to the question
- We explored kinematics in special relativity - motion in a frame of reference - beginning with the Doppler effect for light.

Today

- Briefly discuss the transformation of velocities
- Challenge classical notions of momentum and energy and discuss the implications of new realities for society
- We will begin a discussion of the wave nature of light.

Special Relativistic Transformation of Velocities

One of the most stunning revelations of special relativity proceeds from the vanilla-sounding "transformation of velocities." Recall that the Galilean transformation told us:

$$\text{FRAME } S: \; u = u' + \nu$$

$$\text{FRAME } S': \; u' = u - \nu$$

Special relativity seems to be a better description of the relationship between space and time. What is the "correct" transformation from the perspective of special relativity?

Remember:

$$u = dx/dt$$

and

$$dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial t'}dt'$$

From those relationships, we can derive how the motion of objects in one frame are related to the motions in another frame. For instance, in frame S:

$$u' = \frac{u - \nu}{1 - \frac{\nu u}{c^2}}$$

In Frame S':

$$u = \frac{u' + \nu}{1 + \frac{\nu u'}{c^2}}$$

The Mathematical Bag of Tricks: Binomial Expansion

We keep seeing things that look like $(a + x)^n$, like:

$$\gamma_\nu = \frac{1}{\sqrt{1 - (\nu/c)^2}} \equiv (1 - \beta^2)^{-1/2},$$

or

$$1/\gamma_\nu = (1 - \beta^2)^{+1/2},$$

or in the Doppler Shift

$$(1 + \nu/c)^{-1}.$$

All of these are variations on

$$(a + x)^n.$$

When you see that, and your calculator fails you because $x << 1$, you apply the Binomial Expansion:

$$f(x) = (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ...$$

where $(x^2 < a^2)$

Here are some useful cases:

$$\frac{1}{(1-x^2)} = 1 + x^2 + x^4 + x^6 + ...$$

$$\sqrt{\frac{1}{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + ...$$

This will be helpful for you when, for instance, you are faced with calculating relativistic effects for velocities much smaller than that of light, but where the relativistic effects, though tiny, can have a large impact on the outcome.

Momentum and Energy

Remember the classical conservation of momentum:

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final}$$

What about when you apply the classical transformation of velocities - that is, is momentum conserved in classical physics using the Galilean Transformation?

The answer is yes. If anyone would like to see this, consider a situation where two objects of mass m_1 and m_2 collide head-on. In frame S, their initial momentum and final momentum is given by

$$m_1 u_1^i + m_2 u_2^i = m_1 u_1^f + m_2 u_2^f.$$

In another frame, S', moving along the direction of the collision at speed ν, the form of the conservation equation for the collision should be:

$$m_1 u'^i_1 + m_2 u'^i_2 = m_1 u'^f_1 + m_2 u'^f_2$$

if the conservation of momentum is preserved under different states of motion.

So: are the forms of momentum conservation independent of the motion? We can apply the Galilean Transformation, $u' = u - \nu$, and rewrite the second equation in terms of quantities in frame S. All the terms with ν cancel, and the find that the answer is yes: the statement that classical momentum is invariant under the classical relativistic transformation is true.

But if you instead use the special-relativistic transformation of velocities, you will find the above no longer works. Momentum conservation takes a different form depending on the state of motion of the objects in question relative to an observer.

Discuss what to do next

So what do we do?

- We can throw out momentum conservation as a principle of nature.
- We can change the definition of momentum and find one that is conserved.

Einstein chose to preserve momentum and energy conservation, ala his First Postulate. This was again based on a lack of experimental evidence to the contrary. Instead, he assumed that the form of momentum needed to be amended - that the classical form for momentum, $p = mu$, must be incomplete. So what's the correct form?

The Short Version: Momentum and Energy

Think through the problem of the definition of momentum in class

If $p = mu$ is wrong, what is the right form?

What do we know about momentum as low speeds?

- we know it experimentally takes the form $p = mu$

That means that whatever the form of momentum, $p = \mathcal{M}(u)$, as a function of velocity it must reduce to $p = mu$ for speeds much smaller than that of light.

Do we have a relativistic quantity that behaves this way, which we could multiply times mu?

- What quantity has a minimum value of 1, and is very close to 1 when $u << c$?

The answer is the gamma function, $\frac{1}{\sqrt{1-(\nu/c)^2}}$. So we can postulate that momentum takes the more general form:

$$p = \gamma_u mu$$

, which has the right behavior when $u << c$.

In fact, if you now write down the conservation of momentum equation for this form of the momentum, and correctly transform the velocities from one frame to another, you will find that momentum is conserved again.

Einstein sought then an expression for the total energy of a body in motion. Again, starting from the form of the conservation of Energy equation, he made some arguments about what must happen as a consequence of relativity. In addition, he knew that the energy of motion of a body MUST take the form

$$E = constant + \frac{1}{2}mu^2$$

when $u << c$, since that was what was known from experiments.

What he found was stunning. He found he could express the total energy-squared of a body in motion as

$$E^2 = \left(mc^2\right)^2 + (pc)^2$$

That is, the total energy of a body is related to the sum of its mass and its momentum.

The Long Version: Momentum and Energy

Let's apply Einstein's postulates, specifically:

- The laws of physics are invariant for observers in relative uniform motion

Under that postulate, the form of energy and momentum conservation must be unchanged by motion; in addition, energy and momentum must be conserved in all inertial frames. The classical version of these equations for, say, an elastic collision (as above) is:

$$E_{1b} + E_{2b} = E_{1a} + E_{2a}$$

$$p_{1b} + p_{2b} = p_{1a} + p_{2a}$$

where (classically):

$$p = mu$$

$$E = E(0) + \frac{1}{2}mu^2$$

(the second equation has an additional constant term allowed by Newton's laws:

$$W = \int Fdx = \int (dp/dt)dx = \int m(du/dt)(udt) = \int mudu = \mathcal{C} + \frac{1}{2}mu^2)$$

Let us generalize these ingredients in order to deduce the correct relativistic form:

$$p = \mathcal{M}(u)$$

$$E = \mathcal{E}(u)$$

where $\mathcal{M}(u)$ and $\mathcal{E}(u)$ are unknown functions of the motion of the relative frames. We DO know that in the limit that $u \to 0$:

$$lim_{u\to 0}\mathcal{M}(u) = m$$

$$lim_{u\to 0}\frac{\partial \mathcal{E}(u)}{\partial u^2} = \frac{m}{2}$$

Let's start with momentum. There is so far only one relativistic function that we know which, in the limit $u \rightarrow 0$, goes to 1: $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. We can then guess that:

$$p = \mathcal{M}(u) = \gamma_u m u$$

That leaves us to determine the form of relativistic energy. Energy is trickier because it has a non-linear dependence on velocity of the body in a frame, and it can have a leading constant term in addition to the u^2 term. To get us thinking about how to attack this, let's again think about γ_u. The binomial expansion for γ_u is:

$$\gamma_u = 1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(u^4)$$

We're looking for something that yields, for $u \rightarrow 0$:

$$constant + \frac{1}{2}m\frac{u^2}{c^2}$$

So it seems that we should just try out:

$$\mathcal{E}(u) = \gamma_u \mathcal{E}(0)$$

in the limit of low velocities:

$$lim_{u\rightarrow 0}\mathcal{E}(u) \approx \mathcal{E}(0) + \frac{1}{2}\mathcal{E}(0)\frac{u^2}{c^2}$$

which then leads us to conclude that:

$$\mathcal{E}(0) = mc^2$$

This is incredible. This says that the total energy of a body at rest can be described by its mass. This also tells us that when energy is given off by a body (as in the form of radiation), its mass decreases correspondingly by $\Delta m = \Delta E/c^2$. This is not outside of reason.

- problem: a lightbulb radiates photons, which themselves have no mass, with a power of 75 watts. In one year, how much mass is lost by the lightbulb? Can this be detected using any common measuring instrument?
  - answer: 75 watts means 75 joules/s. A year is $3600s/hour \times 24hours/day \times 365days/year = 3.2 \times 10^7 s$. Let's assume you leave this light on ALL the time. In one year, the lightbulb thus radiates $2.4 \times 10^9 J$, where $1J = 1kg \cdot m^2/s^2$. What is the mass equivalent lost? $\Delta E/c^2 = 2.4 \times 10^9/(3 \times 10^8)^2 = 2.6 \times 10^{-8} kg$. This cannot be measured with any common instrument (a scale, the Wii Fit, etc). Thus, you never would have noticed this mass disappearing, so this idea is not outside of reason because it is not within your experience.

The last equation for energy can be used to find the complete and general expression for the total energy of a body:

$$\mathcal{E}(u)^2 = \gamma_u^2 \mathcal{E}(0)^2 = \gamma_u^2 (mc^2)^2 = \frac{1}{1 - \frac{u^2}{c^2}} m^2 c^4$$

The Binomial Expansion of $1/(1-x^2)$ can be determined using:

$$(1+y)^{-1} = 1 - y + y^2 - y^3 ...$$

where $y = -x^2$:

$$(1-x^2)^{-1} = 1 + x^2 + x^4 + x^6 + ...$$

Plugging into our energy-squared equation:

$$\mathcal{E}(u)^2 = \mathcal{E}(0)^2(1 + (\frac{u}{c})^2 + (\frac{u}{c})^4 + ...$$

$$\mathcal{E}(u)^2 = m^2c^4 + m^2u^2c^2 + m^2u^4 + m^2u^6(1/c^2) + m^2u^8(1/c^4) + ...$$

$$\mathcal{E}(u)^2 = (mc^2)^2 + m^2u^2c^2(1 + u^2/c^2 + u^4/c^4 + u^6/c^6 + ...$$

$$\mathcal{E}(u)^2 = (mc^2)^2 + (\gamma_u mu)^2c^2 = (mc^2)^2 + (pc)^2$$

So the total energy of a body is simply given by the sum of the rest-mass and the momentum of the body. Simple, elegant, and ground-breaking!

At this poiint, we simple redefine $\mathcal{E}(u) \equiv E$ and we have the famous equation:

$$E^2 = m^2c^4 + p^2c^2$$

which describes the total energy of any body in any inertial reference frame.

Consequences of Einstein's *Energy-Momentum-Mass* Revelation

Let's discuss some of the consequences of this innocuous-looking equation:

- What happens to the total energy of a body at rest in a given reference frame?
    - In that case,

      $$E = mc^2$$

      and all of the energy of the body is described by its total mass. Why is this so odd? This equation tells you that if you then heat the object, so that it gains internal energy through temperature increase but NOT kinetic energy through motion, the mass of the object should increase! That is mind-blowing. If you know the mass of something at rest, you know EVERYTHING about its total energy regardless of the form of that energy (light it emits, heat it contains, etc.)
- What happens when the mass of an object is zero? Is that even allowed?

- When mass is zero, $E = pc$. It seems that there is no prior reason to believe such a thing cannot exist; the equation allows it, but what does the equation mean about such an object? If you write this as $E = \gamma_u mu$, the ONLY way that the energy of such an object can be anything but zero is if $u = c$ so that $\gamma_u = \infty$. While relativity fails to tell you about the properties of this object (e.g. exactly what its energy is!), it doesn't rule out that such objects can exist and if they do they must move at the speed of light. What moves at the speed of light? Light. Light is a massless particle, but we won't learn much more about it from relativity since it's undefinable in relativity.

Let's put this together into a question: if a light bulb is just sitting they, emitting light, what is happening to its mass?

- The equation $E = mc^2$ tells us not only that mass is energy, but that when energy is given off by a body (as in the form of radiation), its mass decreases correspondingly by $\Delta m = \Delta E/c^2$. How could this be happening and us not be noticing?
    - Consider a 75W lightbulb. It radiates photons, which themselves have no mass. In one year, how much mass is lost by the lightbulb? Can this be detected using any common measuring instrument? Well, 75 watts means 75 joules/s. A year is $3600s/hour \times 24hours/day \times 365days/year = 3.2 \times 10^7 s$. Let's assume you leave this light on ALL the time. In one year, the lightbulb thus radiates $2.4 \times 10^9 J$, where $1J = 1kg \cdot m^2/s^2$. What is the mass equivalent lost?

      $$\Delta E/c^2 = 2.4 \times 10^9/(3 \times 10^8)^2 = 2.6 \times 10^{-8} kg.$$

      This cannot be measured with any common instrument (a scale, the Wii Fit, etc). Thus, you never would have noticed this mass disappearing, so this idea is not outside of reason because it is not within your experience.

Discuss the implications of changing a little mass into energy

Let's Talk a little bit about light

Relativity includes the speed of light, allows for things like like to exist (as massless objects), but fails to tell us about the properties of light. What is light? To answer that, we'll begin with a discussion about waves.

The Wave Nature of Radiation (Light)

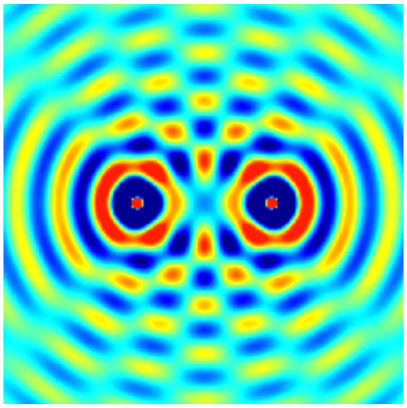
Radiation is energy transmitted from one body to another; for instance, radio waves, light, and heat are all examples of common radiation. The classical understanding of radiation was that it was a wave by nature. That is, radiation (e.g. light) can be described as a phenomenon characterized by a wavelength and a frequency and an amplitude, transmitted at some velocity from one body to another. Waves are continuous phenomena; they deliver arbitrary amounts of energy and that amount of energy is encoded in the intensity of the radiation - more intensity = more energy.

The evidence for the wave-nature of light was multi-fold, and is summarized here:

- Key properties of waves are:
  - diffraction

(from http://en.wikipedia.org/wiki/Diffraction)

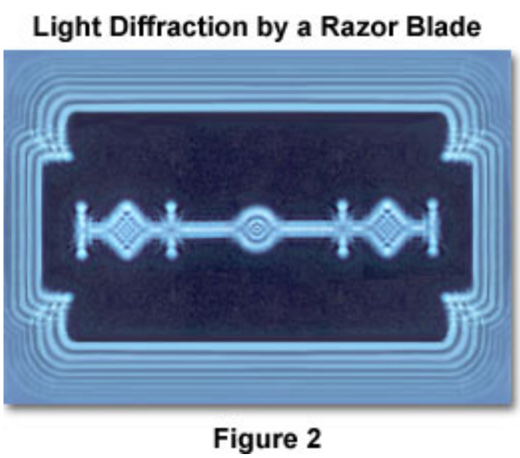
  - interference



(from http://en.wikipedia.org/wiki/Interference_%28wave_propagation%29)

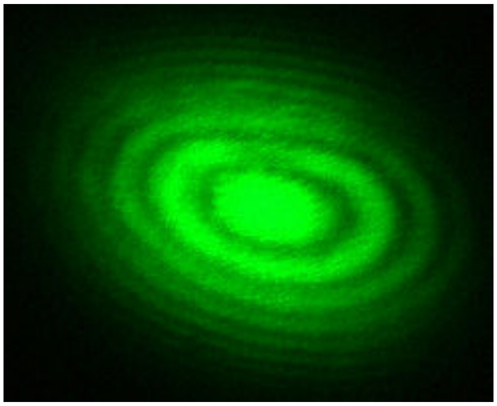
PhET JAVA Demonstration of Wave Properties
(from http://phet.colorado.edu/sims/wave-interference/wave-interference_en.jnlp)

Light was observed to diffract (that is, strike an obstacle and bend around it or, in the case of a small opening, spread out beyond the opening). You can see below an example of diffraction around the edges of a razor blade using blue laser light:



(from http://micro.magnet.fsu.edu/primer/lightandcolor/diffractionintro.html)

Interference of light with itself was also observed. An example below is obtained using a Michelson interferometer (just as in the *Michelson-Morley* experiment), where a single beam of green laser light is split and made to travel two slightly different path lengths, then recombined:



In the mid-1800s, physicists were confident that light/radiation was transmitted via electromagnetic waves.

In addition, classical physics had produced a theory of heat (thermodynamics) that allowed you to predict the spectrum of intensities of heat emitted from a body. Together, they used these two pieces of understanding to study the properties of objects called "blackbodies."

Next time:

We will explore a parallel set of developments that accompanied the Special Theory of Relativity: quantum physics. We will begin with the mystery surrounding black-body radiation and its power spectrum (predicted vs. observed). We will then discuss Max Planck's attempt to resolve that mystery, and the consequences of his hypothesis. We will explore Einstein's contribution to the development of this field, and begin to understand the nature of energy and matter. This will complement our studies of space and time, and will form the foundation upon which the rest of the course will proceed.