# Modern Physics (PHY 3305) Lecture Notes

## Describing Nature as Waves (Ch. 4.3-4.4)

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tags: lecture

**Review**

- We concluded our discussion of light/radiation as particles
- We tried to answer the question: is light a wave or a particle?
  - We discussed duality, the complementarity of the wave-like and particle-like aspected of light
- We asked "why is light special?" and confronted de Broglie's hypothesis: matter has wave and particle behavior, too!
- We looked for evidence of the wave nature of matter (atoms, Bragg scattering)
- We wrote down the wave properties of matter in terms of the particle properties
- We thought about why the wave nature of matter is so undetectable on terrestrial scales

**Matter Waves: Redux**

The de Broglie hypothesis for the wave nature of matter was to simply re-write the equations describing light's dual nature:

$$E_{light} = hf$$

$$p_{light} = h/\lambda$$

and produce the predicted equations for matter:

$$f_{matter} = h/E$$

$$\lambda_{matter} = h/p$$

We also found another connection between the wave and particle aspects of light and matter - that of PROBABILITY. We observed that for light, the probability of finding a photon in some location is proportional to the intensity of the wave-aspect of the light, which is in turn proportional to the square of the amplitude of the wave. Thus

$$P_{photon} \propto A^2_{\mathrm{EMwave}}.$$

This was also observed for electrons (and other matter): the probability of finding one in a given location appeared to be connected to an underlying intensity of some wave, so again

$$P_{matter} \propto A^2_{\mathrm{matterwave}}$$

**Matter at Rest: Infinite Wavelength?**

An outstanding question from last time was:

- Doesn't the wavelength equation for matter suggest that when matter is absolutely still (at rest), its wave nature should be apparent to all because its wavelength is infinite?

This is a very insightful question, and in fact the answer was given in class during the class discussion: is anything ever truly "at rest" - that is, $p = 0$ exactly? This question is going to become very important in the next few lectures. How do we "know" that something is truly at rest, and can something ever truly be exactly at zero motion?

We'll return to this question very soon, but for now consider the fact that I as a person am never truly at rest: my nerves twitch, my cells divide - I am fundamentally in constant motion. So the wave-nature of my matter is held in check by the fact that it's constantly in motion.

**Matter in Motion: Changing the Wave Properties**

Another consequence of these equations lies in the answer to the following question:

- What happens to the wave properties of matter when you increase its speed?

The answer is: the wavelength becomes shorter as the speed increases. Shorter wavelengths mean you can probe smaller and smaller structures in nature, by looking at how matter waves scatter off of targets.

- What are some examples of using matter to probe short distance scales?
  - electron microscope
  - particle accelerators

- If you have to probe small structures, is it better to use light or matter? Why?
  - DISCUSSION

Some ingredients in the discussion:

- visible light microscopes are limits to 100-200 nm resolution due to diffraction in the microscope elements and aberrations in optics.
- you could instead scatter x-rays off of targets
- To obtain an image of the same target (e.g., a 1nm resolution of a 400nm structure), you need light of 1nm or matter of wavelength 1nm.
- Compare the probe particles:
  - Momentum is the same for both: $p = h/\lambda \approx 6.6 \times 10^{-25} J \cdot s/m$
  - The energy of the light (x-rays, given the wavelength) is $E = hf = hc/p \approx 1200 keV\ (2 \times 10^{-16} J)$
    - Light is "hard" to work with because you need good optics to focus the light.
  - The speed of electrons with this momentum: $u \approx p/m = 7.3 \times 10^5 m/s \approx 0.002c$
    - Is that "cheap"? What kind of voltage would you need to apply to get electrons up to that speed? ANSWER: About 1.5V. THAT'S PRETTY CHEAP! Just need a source of electrons (e.g. a hot filament).
    - To focus the electrons just needs magnetic fields, which can be controlled very precisely and in real time (doing this with optics is much harder)

Matter waves are quite useful, since the inherent wavelength of a massive object is already quite small, and charged matter can be controlled (focused) using B fields.

### The Wave Nature of the Cosmos

This fundamental wave behavior of matter and light, which depending on the relevant dimensions of the measurement manifests as either wave or particle behavior, is clearly critical not only in understanding the universe but using matter and energy to better our lives.

If we are going to understand it and apply it, we have to describe it.

This is where The Shroedinger Equation comes in. Before we embark on a discussion of this legendary equation (without the mass appeal of $E = mc^2$, though no less important!), let's go back to the basics of waves.

### Variables of use in wave motion

First of all, let's make some useful definitions of wave variables based off of our existing variables: wavelength and frequency.

DISCUSSION: wave motion

- draw a wave and discuss the cyclic nature of the phenomenon, and how the number of repeats in space and time are valuable information.

Let us quantify this repetitive behavior of a perfect wave. First of all, we have observed that:

- after one wavelength, the wave repeats. Therefore, a useful number is the "wave number" - the spatial frequency of the wave. This is something that is more apparent when you consider describing the wave as a sine function:

  $$A \sin(2\pi x/\lambda).$$

  Notice that when $x = 0$, you are at the zeroth wave. When $x = \lambda$, you've reached the end of the first wave. Wen $x = 2\lambda$, you've reached the end of the second wave. And so on. The wave number can then be written as $k = 2\pi/\lambda$, and indicates at what fraction of the current wave you are located in $x$ (again, this is because the mathematical description of the wave behavior repeats when $x/\lambda$ is an integer). $k$ is the "wave number"

- the spatial frequency of the wave.

- after one cycle of the wave in time, you repeat the cycle. This repetition rate is the frequency. Again, describing the wave in time using a sine function:

  $$A\sin(2\pi ft)$$

  leads to the recognition that the first cycle ends at $t = 1/f$, the second cycle ends at $t = 2/f$, etc. We can then define the "angular frequency" of the wave as $\omega = 2\pi f$, giving you a sense of how long it takes for the wave to conclude one cycle and back to an integer multiple of $2\pi$.

We can then rewrite the particle/wave relationships in terms of these variables:

$$E = h\frac{2\pi}{2\pi}f = \frac{h}{2\pi}\omega \equiv \hbar\omega$$

and similarly,

$$p = \hbar k$$

These are a bit easier to remember, since there is no "dividing by" things in either equation. You just have to remember the definitions of $k$, $\omega$, and $\hbar$.

**The Matter Wave: Amplitude and Probability**

Let us denote the amplitude of the matter wave by

$$\Psi$$

Then the intensity of the wave would be given by

$$|\Psi|^2$$

and thus the probability for finding a particle in a given location where the amplitude of the wave is $\Psi$ is just:

$$P \propto |\Psi|^2$$

**Mechanical Waves**

We now have the pieces to talk about the Shroedinger Wave equation. What is the *wave function* of matter? What is "waving"?

Let's begin by walking through more familiar territory.

As discussed in Homework Problem *SS-2*, the equation for a wave, such as a transverse wave on a stretched string, is:

$$\nu^2 \frac{\partial^2 A(x,t)}{\partial x^2} = \frac{\partial^2 A(x,t)}{\partial t^2}$$

where $\nu$ is the wave speed. The *wave function* here is $A(x,t)$, which tells you the amplitude of the wave as a function of position and time. This equation is derived from a fundamental law - in this case, the mechanical laws of nature regarding forces and acceleration, $\vec{F} = m\vec{a}$.

A basic sinusoidal solution to this equation is the wave function:

$$A(x,t) = |A| \sin(kx - \omega t)$$

where $\omega / k = \nu$.

**Electromagnetic Waves**

We said that the wave nature of light was easy to describe in terms of the oscillating strengths of the electric and magnetic fields that comprise light.

Maxwell's equations can be used to derive the form of the wave equation for light. The solutions to that equation are *PLANE WAVES*, whose properties are:

- They move in one direction at constant velocity
- They do not spread out in space
- They have two parts - an electric oscillation and a magnetic oscillation

The solutions look like:

$$\vec{E}(x,t) = A\sin(kx-\omega t)\hat{y}$$

$$\vec{B}(x,t) = \frac{1}{c}A\sin(kx-\omega t)\hat{z}$$

where $\omega/k = c$.

**Matter Waves**

The wave equation obeyed by matter waves is the Shroedinger Wave Equation. We will for now consider the following case:

- the matter under study is free of the action of external forces - the so-called free-particle case.

In this case, the wave equation is:

$$-\frac{\hbar}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

A few things should be noted:

- $m$ is the mass of the particle
- the equation is complex - that is, the imaginary number $i$ appears, and so the wave functions are not necessarily real - they may be complex functions containing both a real and an imaginary part.
- the equation is second-order in space and first order in time - a differential equation that is somewhat different from the standard mechanical wave equation.

It's form cannot be derived from first principles - it is, in fact, an equation build on the recognition that it makes correct predictions for the outcome of experiments. In that sense, it is a law - one whose predictions are constantly tested by application to experimental situations. It can be motivated, but the form of this equation is not intuitively obvious.

This equation, then, IS a law - like Newton's laws of motion or like

Maxwell's Equations for electromagnetism.

### What is waving

The interpretation of experiments is that PROBABILITY AMPLITUDE is waving - what changes in time and space is not energy or mass, but rather the probability of finding the matter.

### A Discussion of the Complex Nature of the Wave Function

Since the equation contains $i$ and the solutions can be complex, we should discuss the implications of this.

DISCUSSION

- What does it mean when a function is complex? That is, what does it mean to contain imaginary components.
  - it does not mean the function does not represent a real thing; it means that the thing in question cannot simply be described by a real-valued function.
- Revisiting electromagnetism: some treatments consider $\vec{E}$ and $\vec{B}$ as components of a complex vector, $\vec{E} + i\vec{B}$. The mathematical results of this application are the same, even though we appear to be treating the magnetic field as imaginary. Again, it's a representation issue, not a reality issue. Certainly, magnetic fields are REALLY real!
- With the argument about E and B in mind, we've never found a reason to have to treat the real and complex parts of the matter wave function separately, so we treat them as a complex function.
- Does it matter that we can't ascribe "reality" to the wave function itself?
  - DISCUSSION
    - questions to guide the discussion
      - Can you make something measurable from a complex function/number?
      - Are there any examples of how imaginary numbers play a role in the real world? Damping in mechanical and electrical systems.

### Complex Numbers and Functions

Let's begin with a discussion of complex numbers. To kick this off, consider a set of axes in the real number plane:

NEED IMAGE OF REAL NUMBER PLANE AND COMPLEX PLANE

We write a vector in the real number plane as (x,y). When we want to know things about that vector - e.g. its magnitude - we have to compute that as

$$|(x,y)| = \sqrt{x^2+y^2}.$$

As an example, consider the vector in the above plane. What is its magnitude?

We could just as easily represented this vector as a "single-valued number" in the complex plane (see image on right above). Now, we can simply write the number as $z = x + iy$, where $i = \sqrt{-1}$. For the above example, write the complex number.

Now, we want to answer the same question in the complex plane that we answered in the real plane: what is the length of the vector?

- How might I arrive at the magnitude of the complex number?
  - If the answer is to square it, work that through.
  - If the answer is $z^*z$, then do that and use it to explain the complex conjugate, but show why the other case (just squaring it) doesn't work.

There are different ways of writing all of this, which I now introduce:

- Using "real" and "imaginary" parts notation:

$$z = \Re(z) + i\Im(z).$$

    - what are the "real" and "imaginary" parts of the above example?
    - $z^*z$ can then be written:

$$\Re(z)^2 + \Im(z)^2$$

The complex conjugate is a critical ingredient in obtaining a real-valued single number that is the magnitude of the complex number. The same logic works for FUNCTIONS of complex numbers:

Consider a complex function $\Psi$.

We can write this as

$$\Psi = \Re(\Psi) + i\Im(\Psi)$$

Do an example if people want to see one:

Consider the function $f(x) = ax + ibx^2$.

- What are the real and imaginary parts?
- What is the amplitude-squared of the function?

Consider another function:

$$f(x) = ae^{ix}$$

How can I write this in terms of real and imaginary parts?

- You can write the exponential as:

  $$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + ...$$

  - Once you do that, you can separate the series into those parts with and without $i$!

    $$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + .... + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

    - Does anybody recognize these two series? The answer:

      $$e^{ix} = \cos(x) + i\sin(x)$$

      Then it becomes clear how to separate this function into real and imaginary parts.
    - What do you see about the real and imaginary parts of this function? What is their relationship as a function of x?
      - they are 90-degrees (one quarter-cycle) out of phase with one another: $\cos(x) = \sin(x + \pi/2)$

### Back to Probability

We will denote the wave function with $\Psi(x,t)$. We've learned that to compute the amplitude-squared we must take

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t).$$

We've been saying that probability and amplitude-squared are proportional, but to be very clear:

- If the wave function describes a two-dimensional spatial distribution, then the amplitude-squared has units of PROBABILITY PER UNIT LENGTH
- If the wave function described a three-dimensional spatial distribution, then the units are PROBABILITY PER UNIT VOLUME

$$PROBABILITY\ DENSITY = |\Psi(x,t)|^2$$