Problem SS-1: Creating Orthonormal Basis Vectors

Consider:

$$|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

- $$|1\rangle = \frac{1}{\sqrt{\langle I|I\rangle}} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{9}} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

- $$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (0)}_{|0\rangle} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$|2\rangle = \frac{1}{\sqrt{\langle 2'|2'\rangle}} |2'\rangle = \frac{1}{\sqrt{1^2 + 2^2}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \checkmark$$

- $$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle$$

$$= \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \underbrace{\left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right)}_{0}}_{|0\rangle} - \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\left(1\cdot 2 + 2\cdot 5\right)$$

$$= \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}(12) = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{12}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5}\left(\begin{bmatrix} 0 \\ 10 \\ 25 \end{bmatrix} - \begin{bmatrix} 0 \\ 12 \\ 24 \end{bmatrix} \right) = \frac{1}{5}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$|3\rangle = \frac{1}{\sqrt{\langle 3'|3'\rangle}}|3'\rangle = \frac{1}{\sqrt{\frac{1}{25}(4+1)}}\,\frac{1}{5}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \sqrt{5}\,\frac{1}{5}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \checkmark$$

Problem SS-2: Pauli Spin Matrices

1. Demonstrate Unitarity

$$\sigma_x^\dagger \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \checkmark$$

$$\sigma_y^\dagger \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \checkmark$$

$$\sigma_z^\dagger \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\overset{\text{fix}}{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \checkmark$$

2. Demonstrate the relationships $\sigma_i \sigma_j = i\sigma_k$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$$

$$\sigma_y \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x$$

$$\sigma_z \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y$$

* swapping the order of any two will flip the answer by a minus sign.

3. Demonstrate the commutation relations

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i$$

$$= i\epsilon_{ijk}\sigma_k - i\epsilon_{jik}\sigma_k \qquad \epsilon_{ijk} = -\epsilon_{jik}$$

$$= i\epsilon_{ijk}\sigma_k + i\epsilon_{ijk}\sigma_k$$

$$= 2i\epsilon_{ijk}\sigma_k$$

So: $[\sigma_x, \sigma_y] = 2i\sigma_z$ $\quad [\sigma_y, \sigma_z] = 2i\sigma_x$ ✓

$[\sigma_z, \sigma_x] = 2i\sigma_y$

4. Demonstrate the anti-commutation relations:

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 0$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = i\epsilon_{ijk}\sigma_k + i\epsilon_{jik}\sigma_k$$

$$= i\epsilon_{ijk}\sigma_k - i\epsilon_{ijk}\sigma_k$$

$$= \emptyset \checkmark$$

Problem SS-3: Eigenstuff

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

- write the eigenvalue/vector equation:

$$(M - mI)|m\rangle = |0\rangle$$

$$|m\rangle = (M - mI)^{-1}|m\rangle$$

- The determinant must be zero. Thus:

$$\det(M - mI) = 0$$

- we need the determinant of a 3x3 matrix. This is given by:

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} + b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a(ei - fh) + b(di - fg) + c(dh - eg)$$

$$\det(M - mI) = \det\begin{bmatrix} 1-m & 3 & 1 \\ 0 & 2-m & 0 \\ 0 & 1 & 4-m \end{bmatrix}$$

$$= (1-m)\left[(2-m)(4-m)\right] + 3\left[0\right] + 1\left[0\right] = 0$$

The eigenvalues, then, are straight-forward:

$$m_1 = 1$$

$$m_2 = 2$$

$$m_3 = 4$$

- To get the eigenvectors, start with:

$$M|m_1\rangle = m_1|m_1\rangle$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix} = 1 \cdot \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix}$$

$$m_1^1 + 3m_1^2 + m_1^3 = m_1^1$$

$$2m_1^2 = m_1^2 \longrightarrow \boxed{m_1^2 = 0}$$

$$m_1^2 + 4m_1^3 = m_1^3$$

use

$$m_1^1 + m_1^3 = m_1^1$$

$$4m_1^3 = m_1^3 \rightarrow \boxed{m_1^3 = 0}$$

use

$$m_1^1 = m_1^1$$

choose $m_1^1 = 1$ so that $|m_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is automatically <u>unit</u> <u>length</u>.

$$\boxed{|m_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

- second eigenvector :

$$M|m_2\rangle = m|m_2\rangle$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \end{bmatrix} = 2\begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \end{bmatrix}$$

$$m_2^1 + 3m_2^2 + m_2^3 = 2m_2^1$$

$$2m_2^2 = 2m_2^2 \longrightarrow$$ trivial ... we can choose m_2^2 to be anything, so choose $\boxed{m_2^2 = 1}$

$$m_2^2 + 4m_2^3 = 2m_2^3$$

use

$$m_2^1 + 3 + m_2^3 = 2m_2^1$$

$$1 + 4m_2^3 = 2m_2^3 \longrightarrow 2m_2^3 = -1 \longrightarrow \boxed{m_2^3 = -1/2}$$

use

$$m_2^1 + 3 - \frac{1}{2} = 2m_2^1$$

$$m_2^1 = \frac{6}{2} - \frac{1}{2} = \frac{5}{2} \qquad \boxed{m_2^1 = \frac{5}{2}}$$

temporarily:

$$|m_2\rangle = \frac{1}{2}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

normalize

$$|m_2\rangle \longrightarrow \frac{|m_2\rangle}{\sqrt{\langle m_2|m_2\rangle}} = \frac{1}{\sqrt{\frac{1}{4}(25+4+1)}}\,\frac{1}{2}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \frac{2}{\sqrt{30}}\,\frac{1}{2}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{|m_2\rangle = \frac{1}{\sqrt{30}}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}}$$

– Third eigenvector:

$$M|m_3\rangle = m_3|m_3\rangle$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} m_3^1 \\ m_3^2 \\ m_3^3 \end{bmatrix} = 4\begin{bmatrix} m_3^1 \\ m_3^2 \\ m_3^3 \end{bmatrix}$$

$$m_3^1 + 3m_3^2 + m_3^3 = 4m_3^1$$

$$2m_3^2 = 4m_3^2 \longrightarrow \boxed{m_3^2 = 0}$$

$$m_3^2 + 4m_3^3 = 4m_3^3$$

use

$$m_3^1 + m_3^3 = 4m_3^1$$

$$4m_3^3 = 4m_3^3 \longrightarrow \text{trivial} \ldots \text{ set } \boxed{m_3^3 = 1}$$

use

$$m_3^1 + 1 = 4m_3^1$$

$$3m_3^1 = 1$$

$$\boxed{m_3^1 = 1/3}$$

Temporarily:

$$|m_3\rangle = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

normalize:

$$|m_3\rangle \longrightarrow \frac{|m_3\rangle}{\sqrt{\langle m_3|m_3\rangle}} = \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2(1+9)}}\,\frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \frac{3}{\sqrt{10}}\,\frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

so:

$$|m_3\rangle = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\0\\3\end{bmatrix}$$

- solutions:

$$\boxed{\begin{gathered} m_1 = 1, \quad m_2 = 2, \quad m_3 = 4 \\ |m_1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, |m_2\rangle = \begin{bmatrix}5/\sqrt{30}\\2/\sqrt{30}\\-1/\sqrt{30}\end{bmatrix}, |m_3\rangle = \begin{bmatrix}1/\sqrt{10}\\0\\3/\sqrt{10}\end{bmatrix}\end{gathered}}$$