Topics in Spin - Homework 1

March 11, 2014

Problem SS-1: Creating Orthonormal Basis Vectors

Consider the following set of basis vectors,

$$|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},\ |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}.$$

Show how one goes from this set of basis vectors to the *orthonormal* basis vectors,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ |2\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ |3\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Problem SS-2: More on the Pauli Spin Matrices

Consider the Pauli Spin Matrices, which play a central role in spin angular momentum for a spin $-\frac{1}{2}$ system:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Demonstrate that the Pauli Spin Matrices are unitary (that is, that $UU^\dagger = I$, the identity matrix).
2. Demonstrate the relationships among the spin matrices,

   $$\sigma_x\sigma_y = i\sigma_z,\ \sigma_y\sigma_z = i\sigma_x,\ \sigma_z\sigma_x = i\sigma_y.$$

   (This feature is generally known as cyclic permutation - that is, that as long as you preserve the cyclic order i, j, k of the indices (e.g. moving the symbols like they are on a cyclic conveyor belt, such that when one "falls off the end" it returns to the beginning: $i, j, k \rightarrow k, i, j \rightarrow j, k, i$), then it is true that $\sigma_i\sigma_j = i\sigma_k$. The way of enforcing this using mathematical shorthand is the *Levi-Civita Symbol*, ε_{ijk}, which allows us to conveniently write $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$. The feature of this symbol is that $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} = 1$, while $\varepsilon_{ikj} = \varepsilon_{jik} = \varepsilon_{kji} = -1$ (that is, under the swap of any single pair of indices, the function yields -1).

3. Demonstrate the following commutation relations:

$$[\sigma_i, \sigma_j] \equiv \sigma_i \sigma_j - \sigma_j \sigma_i = 2i\sigma_k.$$

4. Demonstrate that the spin matrices *anti-commute*, that is

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 0.$$

Remember these properties - they are immensely useful!

Problem SS-3: Exercise in Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of the following matrix:

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

(HINT: review how to compute the determinant of an $n \times n$ matrix - this is the key to solving eigenvalue/eigenvector problems)