Problem SS-4

Begin with L_+:

$$L_+|\alpha,\beta\rangle = C_+(\alpha,\beta)|\alpha,\beta+\hbar\rangle \rightarrow L_+|\ell,m\rangle = C_+(\ell,m)|\ell,m+1\rangle$$

L_+ is the adjoint of L_-; it must therefore be true that:

$$(L_+|\ell,m\rangle)^\dagger = \langle\ell,m|L_- = (C_+(\ell,m)|\ell,m+1\rangle)^\dagger = \langle\ell,m+1|C_+^*(\ell,m)$$

We can then compute the inner product of the original state, $L_+|\ell,m\rangle$, and its adjoint:

$$\langle\ell,m|L_-L_+|\ell,m\rangle = C_+^*(\ell,m)\,C_+(\ell,m)\underbrace{\langle\ell,m+1|\ell,m+1\rangle}$$

$= 1$, since these are orthonormal basis vectors.

$\downarrow$

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

$\downarrow$

$$\langle\ell,m|(L^2 - L_z^2 - \hbar L_z)|\ell,m\rangle = \langle\ell,m|\left(\ell(\ell+1)\hbar^2 - m^2\hbar^2 - m\hbar^2\right)|\ell,m\rangle$$

$$= \left(\ell(\ell+1) - m^2 - m\right)\hbar^2 \underbrace{\langle\ell,m|\ell,m\rangle}_{1}$$

$$= |C_+(\ell,m)|^2 \underbrace{\langle\ell,m+1|\ell,m+1\rangle}_{1}$$

$$\hbar^2(\ell-m)(\ell+m+1) = |C_+(\ell,m)|^2$$

$$\Rightarrow \boxed{C_+(\ell,m) = \hbar\left[(\ell-m)(\ell+m+1)\right]^{1/2}}$$

Then handle L_-:

$$L_-|l,m\rangle = C_-(l,m)|l,m-1\rangle$$

$$\to \langle l,m|L_+ = \langle l,m-1|\, C_-^*(l,m)$$

$$\langle l,m|L_+L_-|l,m\rangle = \langle l,m|\left(L^2 - L_z^2 + \hbar L_z\right)|l,m\rangle$$

$$= \left[l(l+1)\hbar^2 - m^2\hbar^2 + m\hbar^2\right]\langle l,m|l,m\rangle$$

$$= |C_-(l,m)|^2 \langle l,m-1|l,m-1\rangle$$

$$\hbar^2\left[l(l+1) - m^2 + m\right] = \hbar^2\left[l^2 + l - m^2 + m\right] = \hbar^2(l+m)(l-m+1)$$

$$\boxed{C_-(l,m) = \hbar\left[(l+m)(l-m+1)\right]^{1/2}}$$

$$\boxed{C_\pm(l,m) = \hbar\left[(l\mp m)(l\pm m+1)\right]^{1/2}}$$

Problem SS-5.

Find matrix elements of L^2 for $\ell = 0, 1$.

$$L^2 = \begin{bmatrix} \langle 1|L^2|1\rangle & \langle 1|L^2|2\rangle & \cdots \\ \langle 2|L^2|1\rangle & \ddots & \\ \vdots & & \langle n|L^2|n\rangle \end{bmatrix}$$; here,

$|1\rangle = |\ell=0, m=0\rangle$
$|2\rangle = |\ell=1, m=1\rangle$
$|3\rangle = |\ell=1, m=0\rangle$
$|4\rangle = |\ell=1, m=-1\rangle$

Thus:

$$\langle 1|L^2|1\rangle = \langle 0,0|L^2|0,0\rangle = \emptyset\hbar^2$$

$$\langle 1|L^2|2\rangle = \langle 0,0|L^2|1,1\rangle = \underbrace{1(1+1)}_{2}\hbar^2 \underbrace{\langle 0,0|1,1\rangle}_{\emptyset} = \emptyset\hbar^2$$

$$\langle 2|L^2|2\rangle = \langle 1,1|L^2|1,1\rangle = 2\hbar^2$$

In fact, all off-diagonal elements will be $\emptyset\hbar^2$, while on-diagonal elements can be nonzero.

$$\boxed{L^2 = \begin{bmatrix} 0\hbar^2 & \cdots & & 0 \\ \vdots & 2\hbar^2 & & \vdots \\ & & 2\hbar^2 & \\ 0 & \cdots & & 2\hbar^2 \end{bmatrix}}$$

Next, L_z; let's check a few diagonal and off-diagonal elements:

$$\langle 1|L_z|1\rangle = \langle 0,0|L_z|0,0\rangle = 0\hbar$$

$$\langle 2|L_z|2\rangle = \langle 1,1|L_z|1,1\rangle = \hbar$$

$$\langle 3|L_z|3\rangle = 0\hbar$$

$$\langle 4|L_z|4\rangle = -1\hbar$$

$$\langle 1|L_z|2\rangle = \langle 0,0|L_z|1,1\rangle = 1\hbar \cdot \emptyset = \emptyset\hbar$$

$\vdots$

etc

Again, we find that off-diagonal elements will be zero. (We could have assumed this for L^2 and L_z, since $|l,m\rangle$ are the eigenvectors of <u>both</u> and they commute, so they probably are <u>diagonal</u>.)

$$L_z = \begin{bmatrix} 0\hbar & & & 0 \\ & 1\hbar & & \\ & & 0\hbar & \\ 0 & & & -1\hbar \end{bmatrix}$$

<u>Now for L_x.</u>

$$L_x = \frac{1}{2}(L_+ + L_-)$$

We know that there will be <u>no</u> diagonal elements, since $L_\pm|l,m\rangle = C_\pm(l,m)|l,m\pm1\rangle$, and thus

$$\langle l,m|L_\pm|l,m\rangle$$

will <u>vanish</u>. We need to consider only the $\langle i|L_x|j\rangle$ cases where $\langle i|$ and $|j\rangle$ differ by $\pm1\hbar$ in m.

$$\langle 1|L_x|2\rangle = \langle 0,0|L_x|1,1\rangle = \frac{1}{2}\langle 0,0|(L_+ + L_-)|1,1\rangle$$

$$= \frac{1}{2}\Big(\langle 0,0|C_+(1,1)\underbrace{|0\rangle}_{\text{null ket}} + \underbrace{\langle 0,0|C_-(1,1)|1,0\rangle}_{\text{inner product vanishes due to orthogonality.}}\Big)$$

$$\langle 2|L_x|3\rangle = \langle 1,1|L_x|1,0\rangle$$

$$= \frac{1}{2}\Big(\langle 1,1|L_+|1,0\rangle - \underbrace{\langle 1,1|L_-|1,0\rangle}_{\text{vanishes by orthogonality}}\Big)$$

$$= \frac{1}{2}C_+(1,0)\underbrace{\langle 1,1|1,1\rangle}_{1} = \frac{1}{2}\hbar\left[1\cdot(1+1)\right]^{1/2} = \frac{1}{\sqrt{2}}\hbar$$

$$\langle 3|L_x|2\rangle = \frac{1}{2}\Big(\underbrace{\langle 1,0|L_+|1,1\rangle}_{\substack{|0\rangle \\ \text{null ket}}} + \langle 1,0|L_-|1,1\rangle \Big)$$

$$= \frac{1}{2}\, C_-(1,1)\, \langle 1,0|1,1\rangle$$

$$= \frac{1}{2}\hbar\Big[(1+1)(1)\Big]^{1/2} = \frac{1}{\sqrt{2}}\hbar$$

Thus:

$$L_x = \begin{bmatrix} 0\hbar & 0 & 0 & 0 \\ 0 & 0\hbar & 1/\sqrt{2}\hbar & 0 \\ 0 & 1/\sqrt{2}\hbar & 0\hbar & 1/\sqrt{2}\hbar \\ 0 & 0 & 1/\sqrt{2}\hbar & 0\hbar \end{bmatrix}$$

$\langle 3|L_x|4\rangle \rightarrow$ also $\frac{1}{\sqrt{2}}\hbar$

$\langle 4|L_x|3\rangle$

Finally, L_y:

$$L_y = \frac{-i}{2}\left[L_+ - L_-\right]$$

Again, focus on the off-diagonal elements where $l = l'$ and $m = m^+ \pm 1$:

$$\langle 2|L_y|3\rangle = \frac{-i}{2}\Big(\langle 1,1|(L_+ - L_-)|1,0\rangle\Big) = \frac{-i}{2}\langle 1,1|L_+|1,0\rangle$$

$$= \frac{-i}{2} C_+(1,0) = \frac{-i}{2}(\sqrt{2})\hbar = \frac{-i}{\sqrt{2}}\hbar$$

$$\langle 3|L_y|2\rangle = \frac{-i}{2}\Big(\langle 1,0|(L_+ - L_-)|1,1\rangle\Big) = \frac{-i}{2}\Big(-C_-(1,1)\Big) = \frac{i}{2}\sqrt{2}\,\hbar = \frac{i}{\sqrt{2}}\hbar$$

$$\langle 3|L_y|4\rangle = \frac{-i}{2}\left[\langle 1,0|(L_+ - L_-)|1,-1\rangle\right] = \frac{-i}{2}\langle 1,0|L_+|1,-1\rangle$$

$$= \frac{-i}{2}\, C_+(1,-1) = \frac{-i}{2}\sqrt{(1-(-1))(1-1+1)}\,\hbar$$

$$= \frac{-i}{\sqrt{2}}\hbar$$

$$\langle 4|L_y|3\rangle = \frac{-i}{2}\left[\langle 1,-1|(L_+ - L_-)|1,0\rangle\right] = \frac{-i}{2}\left[-\langle 1,-1|L_-|1,0\rangle\right]$$

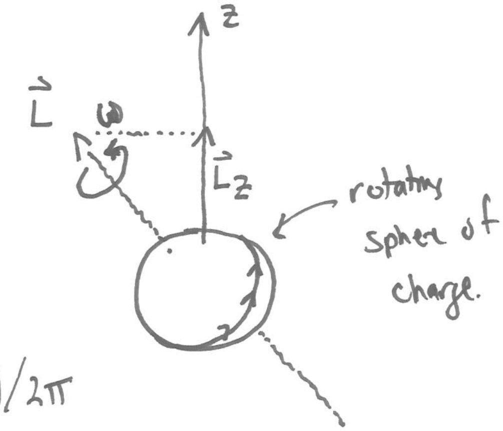
$$= \frac{i}{2}\, C_-(1,0) = \frac{i}{2}\hbar\sqrt{(1+0)(1-0+1)} = \frac{i}{\sqrt{2}}\hbar$$

Thus:

$$L_y = \begin{bmatrix} 0\hbar & 0 & 0 & 0 \\ 0 & 0\hbar & \frac{-i}{\sqrt{2}}\hbar & 0 \\ 0 & \frac{i}{\sqrt{2}}\hbar & 0\hbar & \frac{-i}{\sqrt{2}}\hbar \\ 0 & 0 & \frac{i}{\sqrt{2}}\hbar & 0\hbar \end{bmatrix}$$

Problem SS-6

Classical picture of electron:



$$L_z = \frac{1}{2}\hbar = \frac{1}{2}\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)/2\pi$$

$$L = \sqrt{\frac{3}{4}}\,\hbar$$

$$r_0 = 2.818\times10^{-15}\ \mathrm{m}$$

what is the maximal linear velocity on the sphere?
→ at "equator" of sphere.

$\vec{L} = I\vec{\omega}$. We need the moment of inertia of the sphere. It is unspecified whether this is a shell of charge or a sphere, uniformly populated; Let's consider both:

$$I_{shell} = \frac{2}{3} m r_0^2 \qquad m_e = 9.109\times10^{-31}\ \mathrm{kg}$$

$$I_{solid} = \frac{2 m r_0^2}{5}$$

$$I^e_{shell} = 4.822\times10^{-60}\ \mathrm{kg\cdot m^2}$$

$$I^e_{solid} = 2.893\times10^{-60}\ \mathrm{kg\cdot m^2}$$

$$\omega_{shell} = \frac{|\vec{L}|}{I^e_{shell}} = 1.894\times10^{25}\ \frac{\mathrm{radians}}{\mathrm{s}} \rightarrow V_{equator} = \omega_{shell} r_0 = 5.337\times10^{10}\ \mathrm{m/s}$$

$$\omega_{solid} = \frac{|\vec{L}|}{I^e_{solid}} = 3.156\times10^{25}\ \frac{\mathrm{radians}}{\mathrm{s}} \rightarrow V_{equator} = \omega_{solid} r_0 = 8.895\times10^{10}\ \mathrm{m/s}$$

Both exceed the speed of light. The charge radius of the electron in modern experiments is known to be $<10^{-18}$ m. This would make the electron have to spin even faster.