SS-7: Eigenstates of S_x, S_y

We begin by writing:

$$S_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we need to solve the eigenvalue problem for both.

$$\det(S_x - Im) = 0 \Rightarrow \det\left(\frac{1}{2}\hbar\begin{pmatrix} -\frac{2}{\hbar}m & 1 \\ 1 & -\frac{2m}{\hbar} \end{pmatrix}\right) = 0$$

$$\Rightarrow \frac{4}{\hbar^2}m^2 - 1 = 0$$

$$m^2 = \frac{\hbar^2}{4} \qquad \boxed{m = \pm\frac{\hbar}{2}}$$

now, use $m = +\hbar/2$:

$$S_x|\psi_1\rangle = \frac{\hbar}{2}|\psi_1\rangle \Rightarrow \frac{1}{\cancel{2}}\cancel{\hbar}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \cancel{\frac{1}{2}\hbar}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = a$$
$$a = b$$

choose $a = 1$, $b = 1$:

$$|\psi_1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xRightarrow{\text{normalize}} \boxed{|\psi_1'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

now for $m = -\hbar/2$

$$\frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{2}\hbar\begin{bmatrix} a \\ b \end{bmatrix}$$

$$b = -a$$
$$a = -b$$

$\Rightarrow$ choose $a = 1$, $b = -1$

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then: $|\psi_2\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow[\text{normalize}]{} \boxed{|\psi_2'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

Thus the eigenstates of S_x are:

$$\boxed{|\psi_x(1)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |\psi_x(2)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

check: $\langle \psi_x(2)|\psi_x(1)\rangle = \frac{1}{2}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \emptyset$ ✓

Now for S_y:

$$\det(S_y - mI) = \emptyset$$

$$\det\left[\begin{pmatrix} 0 & -\frac{i}{2}\hbar \\ \frac{i}{2}\hbar & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right] = \emptyset$$

$$\det\begin{pmatrix} -m & -\frac{i}{2}\hbar \\ \frac{i}{2}\hbar & -m \end{pmatrix} = \emptyset$$

$$m^2 - \left(\frac{1}{4}\hbar^2\right) = \emptyset$$

$$\boxed{m = \pm\frac{\hbar}{2}}$$ ✓

then choose $m = +\hbar/2$

fix!

↓

$$\cancel{\tfrac{1}{2}}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \cancel{\tfrac{\hbar}{2}}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{array}{l} -ib = \frac{\hbar}{2}a \\ ia = b \Rightarrow a = -ib \end{array}$$ ✓

Choose: $\boxed{a=1,\ b=i}$

thus $|\psi_1\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \xRightarrow[\text{normalize}]{} \boxed{|\psi_y(1)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}}$

<u>then choose $m = -\frac{\hbar}{2}$</u>

$$\cancel{\frac{1}{2}\hbar}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\cancel{\frac{1}{2}\hbar}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$-ib = -a \Rightarrow ib = a$$

$$ia = -b \Rightarrow -a = -ib \Rightarrow a = ib$$

so, choose $\boxed{a=1,\ b=-i}$

$|\psi_2\rangle = \begin{bmatrix} 1 \\ -i \end{bmatrix} \xRightarrow[\text{normalize}]{} \boxed{|\psi_y(2)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}}$

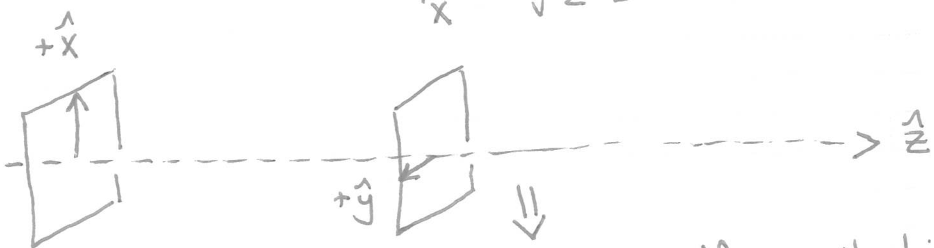
check:

$$\langle \psi_y(2)|\psi_y(1)\rangle = \frac{1}{2}[1 \ \ +i]\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2}(1-1) = \emptyset \checkmark$$

Part 2:

We prepare an electron in a pure eigenstate of S_x, along $+\hat{x}$:

$$|\psi_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$



$\Downarrow$

what is the probability of finding the spin to be along $+\hat{y}$?

$$\langle\psi_y(1)|\psi_x\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}(1-i)$$

= a complex number, z.

$$P = |\langle w|v\rangle|^2 = z^* z = \frac{1}{2}\cdot\frac{1}{2}(1+i)(1-i) = \frac{1}{4}(1-i+i+1)$$
$$= \frac{2}{4} = \frac{1}{2}$$

$P = 50\% \Longrightarrow$ there is a 50% chance of finding the spin to be projected along $+\hat{y}$.

Part 3

What are $\langle S_x\rangle$ and $\langle S_y\rangle$ ~~of~~ for the originally prepared state?

$$\langle S_x\rangle = \langle\psi_x|S_x|\psi_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{2}\hbar\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$
$$= \frac{1}{2}\cdot\frac{1}{2}\hbar\left[1^2+1^2\right] = \frac{2}{4}\hbar = \frac{1}{2}\hbar$$

this makes sense, as $|\psi_x\rangle$ was an eigenstate of S_x.

what about $\langle S_y \rangle$?

$$\langle \psi_x | S_y | \psi_x \rangle = \frac{1}{\sqrt{2}} [1 \ 1] \ \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \cdot \frac{1}{2}\hbar \ [1 \ 1] \begin{bmatrix} -i \\ i \end{bmatrix} = \emptyset$$

This also makes sense – the prepared state was an eigenstate of S_x – we expect that $\langle S_y \rangle = \emptyset$ as a result

<u>part 4</u> : what is going on?

Even though ψ was prepared in an eigenstate of S_x, with $\langle S_x \rangle = \frac{1}{2}\hbar$ and $\langle S_y \rangle = \emptyset$, we do not know <u>exactly</u> where ψ is pointing in x any y because S_x and S_y are non-commuting – they have an uncertainty principle that applies to both, $\Delta S_x \, \Delta S_y \geq \hbar/2$. $\langle S_x \rangle$ and $\langle S_y \rangle$ are only <u>expectation values</u> – what we typically expect to observe over an ensemble of measurements. But in reality, there is some fraction of $\vec{S}$ that points along $\hat{y}$ at any given time. Thus there is a probability (50%) that if we pass ψ through a filter along $\hat{y}$, we will find that $\vec{S}$ points along $\hat{y}$.

SS-8

Show that, for a two-particle system,

$$S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$\vec{S} = \vec{S}_1 + \vec{S}_2$

$$S_{1+} = S_{1x} + iS_{1y} \qquad S_{1-} = S_{1x} - iS_{1y}$$

$$S_{2+} = S_{2x} + S_{2y} \qquad S_{2-} = S_{2x} - iS_{2y}$$

also:

$$S_{1+}S_{2-} = S_{1x}S_{2x} - iS_{1x}S_{2y} + iS_{1y}S_{2x} + S_{1y}S_{2y}$$

$$S_{1-}S_{2+} = S_{1x}S_{2x} + iS_{1x}S_{2y} - iS_{1y}S_{2x} + S_{1y}S_{2y}$$

now:

$$S^2 = (\vec{S}_1 + \vec{S}_2)(\vec{S}_1 + \vec{S}_2) = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$

$$= S_1^2 + S_2^2 + 2(\underbrace{S_{1x}S_{2x} + S_{1y}S_{2y}} + S_{1z}S_{2z})$$

consider this piece:

$$2S_{1x}S_{2x} + 2S_{1y}S_{2y} = 2S_{1x}S_{2x} + [iS_{1x}S_{2y} - iS_{1x}S_{2y} + iS_{1y}S_{2x} - iS_{1y}S_{2x}] + 2S_{1y}S_{2y}$$

$$= S_{1x}S_{2x} - iS_{1x}S_{2y} + iS_{1y}S_{2x} + S_{1y}S_{2y} + S_{1x}S_{2x} + iS_{1x}S_{2y} - iS_{1y}S_{2x} + S_{1y}S_{2y}$$

$$= S_{1+}S_{2-} + S_{1-}S_{2+}$$

thus:

$$S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+} \quad \checkmark$$

Show that the middle block of S^2 are all $1\hbar^2$.

we need to compute:

$$\langle j | S^2 | i \rangle$$

where in this case $|i\rangle = |+-\rangle$ or $|-+\rangle$. Label the states:

$$|1\rangle = |+-\rangle$$
$$|2\rangle = |-+\rangle$$

compute the central block:

$$\langle 1 | S^2 | 1 \rangle = \langle +- | S^2 | +- \rangle$$
$$= \langle +- | \left(S_1^2 + S_2^2 + 2 S_{1z} S_{2z} + S_{1+} S_{2-} + S_{1-} S_{2+} \right) | +- \rangle$$
$$= \langle +- | \Big(\left[s_1(s_1+1)\hbar^2 + s_2(s_2+1)\hbar^2 + 2 m_1 \hbar m_2 \hbar \right] |+-\rangle + \left[C_- C_+ |-+\rangle \right] \Big)$$
$$= \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 - \frac{2}{4}\hbar^2 = 1\hbar^2$$

Similarly:

$$\langle 2 | S^2 | 2 \rangle = \langle -+ | S^2 | -+ \rangle$$
$$= \langle -+ | \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 - \frac{2}{4}\hbar^2 \right) | -+ \rangle + C_+ C_- \underbrace{\langle -+ | +- \rangle}_{\emptyset}$$
$$= 1\hbar^2$$

Now for the off-diagonal elements:

$$\langle 1|S^2|2\rangle = \langle +-|S^2|-+\rangle$$

$$= \langle +-|\left[\hbar^2|-+\rangle + C_- C_+|-+\rangle\right]$$

$$= C_- C_+ = C_-\left(\tfrac{1}{2},\tfrac{1}{2}\right) C_+\left(\tfrac{1}{2},-\tfrac{1}{2}\right)$$

$$= \hbar\sqrt{1(1)}\;\hbar\sqrt{1(1)} = \hbar^2$$

$$\langle 2|S^2|1\rangle = \langle -+|S^2|+-\rangle$$

$$= C_+ C_- = \hbar^2$$

So the block is:

$$S^2_{i=1,2;\, j=1,2} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \hbar^2$$

This creates off-diagonal elements in the center of the S^2 matrix in this basis. These are <u>not</u> eigenstates

Problem SS-9: 3-spin-1/2 particle system

We begin by writing the kets describing the possible orientations of the 3 z-projections of the spin. For each particle,

$$S_i^2 |\psi\rangle = \underbrace{s_i(s_i+1)}_{3/4}\hbar^2 |\psi\rangle = \frac{3}{4}\hbar^2 |\psi\rangle$$

$$s_i \rightarrow \frac{1}{2}, \quad s_i \rightarrow \frac{1}{2}$$

We need only label the z-projections:

$$S_{z_i} |\psi\rangle = \pm\frac{1}{2}\hbar |\psi\rangle$$

For instance, when all spins point up along $\hat{z}$:

$$|+\,+\,+\rangle$$

Let us label this as $|1\rangle = |+++\rangle$. We can write the other 7 "unique" kets:

all up: $|+++\rangle$
one up: $|+--\rangle$, $|-+-\rangle$, $|--+\rangle$
two up: $|++-\rangle$, $|+-+\rangle$, $|--+\rangle$
all down: $|---\rangle$

Let us write:

$$\begin{aligned}
|1\rangle &= |+++\rangle \\
|2\rangle &= |++-\rangle \\
|3\rangle &= |+-+\rangle \\
|4\rangle &= |-++\rangle \\
|5\rangle &= |--+\rangle \\
|6\rangle &= |-+-\rangle \\
|7\rangle &= |+--\rangle \\
|8\rangle &= |---\rangle
\end{aligned}$$

We can now write the expressions for the S^2 and S_z operators.

$$S_z = S_{1z} + S_{2z} + S_{3z}$$

$$\begin{aligned} S^2 = {} & S_1^2 + S_2^2 + S_3^2 + \\ & 2S_{1z}S_{2z} + 2S_{1z}S_{3z} + 2S_{2z}S_{3z} + \\ & S_{1+}S_{2-} + S_{1-}S_{2+} + \\ & S_{1+}S_{3-} + S_{1-}S_{3+} + \\ & S_{2+}S_{3-} + S_{2-}S_{3+} \end{aligned}$$

modelled on what we learned from the 2-particle case.

By this point, I hope it is clear that

$$\langle j | S_{iz} | k \rangle = m_{si}\hbar\, \delta_{jk} \quad \text{(a \underline{diagonal} matrix)}$$

and that

$$\langle j | S_i^2 | k \rangle = s_i(s_i+1)\hbar^2\, \delta_{jk} \quad \text{(also \underline{diagonal})}$$

so that also:

$$\langle j | S_{az} S_{bz} | k \rangle = m_{sa} m_{sb} \hbar^2\, \delta_{jk} \quad \text{(diagonal)}$$

we can separate out the parts of S_z, S^2 that <u>will</u> be diagonal and write each piece.

$$S_z = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 3/2 & & & & & & & \\ 2 & & 1/2 & & & & & & \\ 3 & & & 1/2 & & & & & \\ 4 & & & & 1/2 & & & & \\ 5 & & & & & -1/2 & & & \\ 6 & & & & & & -1/2 & & \\ 7 & & & & & & & -1/2 & \\ 8 & & & & & & & & -3/2 \end{array} \ \hbar \quad \Rightarrow \text{ move on to } S^2$$

$$S_i^2 = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 3/4 &&&&&&& \\ & 3/4 &&&&&& \\ && 3/4 &&&&& \\ &&& 3/4 &&&& \\ &&&& 3/4 &&& \\ &&&&& 3/4 && \\ &&&&&& 3/4 & \\ &&&&&&& 3/4 \end{bmatrix} \end{array} \hbar^2 \Longrightarrow \boxed{\sum_{i=1}^{3} (S_i^2)_{jk} = \frac{9}{4}\hbar^2 \delta_{jk}} \quad \forall j,k = 1, 2, \dots 8$$

Now for the z-projection pieces:

$$2S_{1z}S_{2z} = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 2/4 &&&&&&& \\ & 2/4 &&&&&& \\ && -2/4 &&&&& \\ &&& -2/4 &&&& \\ &&&& 2/4 &&& \\ &&&&& -2/4 && \\ &&&&&& -2/4 & \\ &&&&&&& 2/4 \end{bmatrix} \end{array} \hbar^2$$

$$2S_{1z}S_{3z} = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 2/4 &&&&&&& \\ & -2/4 &&&&&& \\ && 2/4 &&&&& \\ &&& -2/4 &&&& \\ &&&& -2/4 &&& \\ &&&&& 2/4 && \\ &&&&&& -2/4 & \\ &&&&&&& 2/4 \end{bmatrix} \end{array} \hbar^2$$

$$2S_{2z}S_{3z} = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 2/4 &&&&&&& \\ & -2/4 &&&&&& \\ && -2/4 &&&&& \\ &&& 2/4 &&&& \\ &&&& -2/4 &&& \\ &&&&& -2/4 && \\ &&&&&& 2/4 & \\ &&&&&&& 2/4 \end{bmatrix} \end{array} \hbar^2$$

Thus:

$$2(S_{1z}S_{2z} + S_{1z}S_{3z} + S_{2z}S_{3z}) = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 6/4 &&&&&&& \\ & -2/4 &&&&&& \\ && -2/4 &&&&& \\ &&& -2/4 &&&& \\ &&&& -2/4 &&& \\ &&&&& -2/4 && \\ &&&&&& -2/4 & \\ &&&&&&& 6/4 \end{bmatrix} \end{array} \hbar^2$$

Now we can deal with the asymmetric parts.

Let us begin with the term: $S_{1+}S_{2-} + S_{1-}S_{2+}$:

$$\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right)|1\rangle = |0\rangle \quad \text{(null ket)}$$

$$\begin{aligned}
&'' \quad |2\rangle = |0\rangle \\
&'' \quad |3\rangle = K|4\rangle \qquad K = C_+\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)C_-\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \\
&'' \quad |4\rangle = K|3\rangle \qquad = \hbar^2 \\
&'' \quad |5\rangle = |0\rangle \\
&'' \quad |6\rangle = K|7\rangle \\
&'' \quad |7\rangle = K|6\rangle \\
&'' \quad |8\rangle = |0\rangle
\end{aligned}$$

So we see that this matrix will only have non-zero elements <u>off the</u> <u>diagonal</u> at the locations (3,4), (4,3), (6,7), and (7,6).

Then we have:

$$\left(S_{1+}S_{3-} + S_{1-}S_{3+}\right)|1\rangle = |0\rangle$$

$$\begin{aligned}
|2\rangle &= K|4\rangle \\
|3\rangle &= |0\rangle \\
|4\rangle &= K|2\rangle \\
|5\rangle &= K|7\rangle \\
|6\rangle &= |0\rangle \\
|7\rangle &= K|5\rangle \\
|8\rangle &= |0\rangle
\end{aligned}$$

with off-diagonal nonzero elements at (2,4), (4,2), (5,7), and (7,5)

Finally, we have:

$$\left(S_{2+}S_{3-} + S_{2-}S_{3+}\right)|1\rangle = |0\rangle$$

$$\text{"} \qquad |2\rangle = K|3\rangle$$

$$\text{"} \qquad |3\rangle = K|2\rangle$$

$$\text{"} \qquad |4\rangle = |0\rangle$$

$$\text{"} \qquad |5\rangle = K|6\rangle$$

$$\text{"} \qquad |6\rangle = K|7\rangle$$

$$\text{"} \qquad |7\rangle = |0\rangle$$

$$\text{"} \qquad |8\rangle = |0\rangle$$

with non-zero off-diagonal elements at (2,3), (3,2), (6,7), and (7,6).

So we can write the matrix as:

$$S_{1+}S_{2-} + S_{1-}S_{2+} + S_{1+}S_{3-} + S_{1-}S_{3+} + S_{2+}S_{3-} + S_{2-}S_{3+} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} & & & & & & & \\ & & 1 & 1 & & & & \\ & 1 & & 1 & & & & \\ & 1 & 1 & & & & & \\ & & & & & 1 & 1 & \\ & & & & 1 & & 1 & \\ & & & & 1 & 1 & & \\ & & & & & & & \end{bmatrix} \end{array} \hbar^2$$

The full S^2 matrix is then:

$$S^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 15/4 & & & & & & & \\ & 7/4 & 1 & 1 & & & & \\ & 1 & 7/4 & 1 & & & & \\ & 1 & 1 & 7/4 & & & & \\ & & & & 7/4 & 1 & 1 & \\ & & & & 1 & 7/4 & 1 & \\ & & & & 1 & 1 & 7/4 & \\ & & & & & & & 15/4 \end{bmatrix} \end{array} \hbar^2$$

Which is not diagonal. But we can diagonalize it to find the basis states of S^2.

I used OCTAVE to do this. See the attached example program.

The result is a diagonalized S^2

$$S^2 = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 3/4 & & & & & & & \\ & 3/4 & & & & & & \\ & & 3/4 & & & & & \\ & & & 3/4 & & & & \\ & & & & 15/4 & & & \\ & & & & & 15/4 & & \\ & & & & & & 15/4 & \\ & & & & & & & 15/4 \end{bmatrix} \end{array} \hbar^2$$

with 2 distinct total spin quantum numbers:

$$S = \tfrac{1}{2}, \quad S^2 |\psi\rangle = \tfrac{3}{4}\hbar^2 |\psi\rangle \rightarrow \text{spin-}1/2$$

$$S = 3/2, \quad S^2 |\psi\rangle = \tfrac{15}{4}\hbar^2 |\psi\rangle \rightarrow \text{spin-}3/2$$

The eigen vectors are now:

$$|1\rangle = \frac{1}{\sqrt{6}}\left(|++-\rangle + |+-+\rangle - 2|-++\rangle \right)$$

$$|2\rangle = \frac{1}{\sqrt{6}}\left(|--+\rangle + |-+-\rangle - 2|+--\rangle \right)$$

$$|3\rangle = \frac{1}{\sqrt{2}}\left(|--+\rangle - |-+-\rangle \right)$$

$$|4\rangle = \frac{1}{\sqrt{2}}\left(|++-\rangle - |+-+\rangle \right)$$

$$|5\rangle = \frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle\right)$$

$$|6\rangle = \frac{1}{\sqrt{3}}\left(|--+\rangle + |-+-\rangle + |+--\rangle\right)$$

$$|7\rangle = |+++\rangle$$

$$|8\rangle = |---\rangle$$

What about S_z?

$S_z|1\rangle = +\frac{1}{2}\hbar|1\rangle \qquad (s = 1/2)$

$S_z|2\rangle = -\frac{1}{2}\hbar|2\rangle \qquad (s = 1/2)$

$S_z|3\rangle = -\frac{1}{2}\hbar|3\rangle \qquad (s = 1/2)$

$S_z|4\rangle = +\frac{1}{2}\hbar|4\rangle \qquad (s = 1/2)$

$S_z|5\rangle = +\frac{1}{2}\hbar|5\rangle \qquad (s = 3/2)$

$S_z|6\rangle = -\frac{1}{2}\hbar|6\rangle \qquad (s = 3/2)$

$S_z|7\rangle = +\frac{3}{2}\hbar|7\rangle \qquad (s = 3/2)$

$S_z|8\rangle = -\frac{3}{2}\hbar|8\rangle \qquad (s = 3/2)$

Also diagonal. We have found the common eigenbasis!

What is our spectrum of states? We have 2 doublets ($s=\frac{1}{2}$, $m_s = \pm\frac{1}{2}$) and one quartet ($s=\frac{3}{2}$, $m_s = \pm\frac{3}{2}, \pm\frac{1}{2}$) of states.

The proton cannot be represented by the quartet of states – they have the wrong total spin quantum number, $s = 3/2$. It has to be one of the doublets, since they both have spin-$1/2$ ($s = 1/2$).

```
octave:3> S2=[15/4,0,0,0,0,0,0,0;0,7/4,1,1,0,0,0,0;0,1,7/4,1,0,0,0,0;0,1,1,7/4,0,0,0,0;0,0,0,0,7/4,1,1,0;0,0,0,0,1,7/4,1,0;0,0,0,0,1,1,7/4,0;0,
0,0,0,0,0,0,15/4]
S2 =

   3.75000   0.00000   0.00000   0.00000   0.00000   0.00000   0.00000   0.00000
   0.00000   1.75000   1.00000   1.00000   0.00000   0.00000   0.00000   0.00000
   0.00000   1.00000   1.75000   1.00000   0.00000   0.00000   0.00000   0.00000
   0.00000   1.00000   1.00000   1.75000   0.00000   0.00000   0.00000   0.00000
   0.00000   0.00000   0.00000   0.00000   1.75000   1.00000   1.00000   0.00000
   0.00000   0.00000   0.00000   0.00000   1.00000   1.75000   1.00000   0.00000
   0.00000   0.00000   0.00000   0.00000   1.00000   1.00000   1.75000   0.00000
   0.00000   0.00000   0.00000   0.00000   0.00000   0.00000   0.00000   3.75000

octave:4> [EVECT,EVAL]=eig(S2)
EVECT =

   0.00000   0.00000  -0.00000  -0.00000   0.00000   0.00000   1.00000   0.00000
   0.40825   0.00000  -0.00000   0.70711   0.57735   0.00000   0.00000   0.00000
   0.40825   0.00000  -0.00000  -0.70711   0.57735   0.00000   0.00000   0.00000
  -0.81650   0.00000  -0.00000   0.00000   0.57735   0.00000   0.00000   0.00000
  -0.00000   0.40825   0.70711   0.00000   0.00000   0.57735   0.00000   0.00000
  -0.00000   0.40825  -0.70711   0.00000   0.00000   0.57735   0.00000   0.00000
  -0.00000  -0.81650   0.00000   0.00000   0.00000   0.57735   0.00000   0.00000
  -0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000   0.00000   1.00000

EVAL =

Diagonal Matrix

   0.75000         0         0         0         0         0         0         0
         0   0.75000         0         0         0         0         0         0
         0         0   0.75000         0         0         0         0         0
         0         0         0   0.75000         0         0         0         0
         0         0         0         0   3.75000         0         0         0
         0         0         0         0         0   3.75000         0         0
         0         0         0         0         0         0   3.75000         0
         0         0         0         0         0         0         0   3.75000

octave:5> 1/sqrt(6)
ans =  0.40825
octave:6> -0.81650/0.40825
ans = -2
octave:7> sqrt(2)
ans =  1.4142
octave:8> 1/sqrt(2)
ans =  0.70711
octave:9> 1/0.57735^2
ans =  3.0000
octave:10>
```