In-Class Exercise: Practice with Vectors...

1) Compute the complex conjugates

- begin with $|I\rangle = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$
- transpose the vector: $(|I\rangle)^T = [1 \quad 5]$
- take the complex conjugate: $(|I\rangle)^{T*} = [1^* \quad 5^*] = [1 \quad 5]$

$= \langle I|$

- repeat for $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

$$(|II\rangle)^{T*} \equiv \langle II| = [7 \quad 0]$$

2) Compute the lengths

- $|I| = \sqrt{\langle I|I\rangle} = \sqrt{[1 \quad 5]\begin{bmatrix} 1 \\ 5 \end{bmatrix}} = \sqrt{1^2 + 5^2}$

$= \sqrt{26}$

- $|II| = \sqrt{\langle II|II\rangle} = \sqrt{[7 \quad 0]\begin{bmatrix} 7 \\ 0 \end{bmatrix}} = \sqrt{49} = 7$

(which, in this case, can be determined by inspection.)

3) Demonstrate that they are non-parallel

- Compute the inner product:

$$\langle II|I\rangle = [7 \quad 0]\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 7 \cdot 1 + 0 \cdot 5 = 7$$

$\neq 0$

$\therefore$ non-parallel

- Check Schwartz Inequality – if $\underbrace{|\langle II|I\rangle|}_{7} < \underbrace{|II||I|}_{7 \cdot \sqrt{26}}$, then non-parallel. ✓

4) Find an orthonormal basis vector set.

- Begin with $|I\rangle$ and apply the Gram-Schmidt theorem to develop the basis vectors:

$$|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{\sqrt{26}}\begin{bmatrix}1\\5\end{bmatrix}$$

verify unit normalization:

$$\langle 1|1\rangle = \left(\frac{1}{\sqrt{26}}\right)^2 \begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}$$

$$= \frac{1}{26}(1+25) = 1 \checkmark$$

- Determine $|2\rangle$ from $|1\rangle$ and $|II\rangle$

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

$$= \begin{bmatrix}7\\0\end{bmatrix} - \begin{bmatrix}1/\sqrt{26}\\5/\sqrt{26}\end{bmatrix}\begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}7\\0\end{bmatrix}\frac{1}{\sqrt{26}}$$

$$= \begin{bmatrix}7\\0\end{bmatrix} - \left(\frac{1}{\sqrt{26}}\right)^2\begin{bmatrix}1\\5\end{bmatrix}(7)$$

$$= \begin{bmatrix}7 - 7/26\\0 - 35/26\end{bmatrix}$$

$$= \begin{bmatrix}7(26/26 - 1/26)\\-35/26\end{bmatrix} = \frac{1}{26}\begin{bmatrix}7\cdot 25\\-35\end{bmatrix}$$

$$= \frac{5}{26}\begin{bmatrix}35\\-7\end{bmatrix} = \frac{35}{26}\begin{bmatrix}5\\-1\end{bmatrix}$$

$$|2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\frac{35\sqrt{26}}{26}} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad \frac{35}{26}$$

$$\boxed{\begin{aligned} \langle 2'|2'\rangle &= \left(\frac{35}{26}\right)^2 (25+1) \\ &= \left(\frac{35}{26}\right)^2 (26) \\ \sqrt{\langle 2'|2'\rangle} &= \frac{35}{26}\sqrt{26} = \frac{35}{\sqrt{26}} \end{aligned}}$$

$$= \frac{\sqrt{26}}{\cancel{35}} \frac{\cancel{35}}{26} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

- so we have:

$$|1\rangle = \frac{1}{\sqrt{26}} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$|2\rangle = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

- are they orthogonal? Check:

$$\langle 2|1\rangle = \frac{1}{26} \begin{bmatrix} 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$= \frac{1}{26}(5-5) = 0 \quad \checkmark$$

In-Class Exercise: The Rotation Operator, R

To determine the matrix elements, we begin with the actions of the operator on the basis vectors:

$$R\left(\tfrac{1}{2}\pi\vec{i}\right)|i\rangle = |i\rangle$$

$$R\left(\tfrac{1}{2}\pi\vec{i}\right)|j\rangle = |k\rangle$$

$$R\left(\tfrac{1}{2}\pi\vec{i}\right)|k\rangle = -|j\rangle$$

The schematic for determining the elements R_{ij} is:

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} \langle 1|R|1\rangle & \langle 1|R|2\rangle & \langle 1|R|3\rangle \\ \langle 2|R|1\rangle & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where $|1\rangle = |i\rangle$, $|2\rangle = |j\rangle$, and $|3\rangle = |k\rangle$. Look at these:

$$\langle 1|R|1\rangle = \langle i|\underbrace{R|i\rangle}_{|i\rangle} = \langle i|i\rangle = 1$$

$$\langle 1|R|2\rangle = \langle 1|\underbrace{R|j\rangle}_{|k\rangle} = \langle i|k\rangle = 0$$

fix

$$\langle 1|R|3\rangle = \langle i|R|k\rangle = \langle i|(-|j\rangle) = -\langle i|j\rangle = 0$$

$$\langle 2|R|1\rangle = \langle j|R|i\rangle = \langle j|i\rangle = 0$$

$$\langle 2|R|2\rangle = \langle j|R|j\rangle = \langle j|k\rangle = 0$$

$$\langle 2|R|3\rangle = \langle j|R|k\rangle = -\langle j|j\rangle = -1$$

$$\langle 3|R|1\rangle = \langle k|R|i\rangle = \langle k|i\rangle = 0$$

$$\langle 3|R|2\rangle = \langle k|R|j\rangle = \langle k|k\rangle = 1$$

$$\langle 3|R|3\rangle = \langle k|R|k\rangle = -\langle k|j\rangle = 0$$

We can now write the matrix:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

verify it does these rotations:

$$R|i\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = |i\rangle \quad \checkmark$$

$$R|j\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |k\rangle \quad \checkmark$$

$$R|k\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -|j\rangle \quad \checkmark$$