In-Class Exercise: momentum of plane wave

[in the original lecture notes, $\hbar$ was used in the exponent instead of $\hbar^{-1}$. The $\underline{2\pi}$ was also missing]

$$|\psi(x,y,z,t)\rangle = Ae^{-\frac{i}{\hbar}(kx-\omega t)2\pi}$$

Act with the momentum operator, p_x:

$$p_x|\psi\rangle = -i\hbar\left(\frac{\partial}{\partial x}\right)Ae^{-\frac{i}{\hbar}(kx-\omega t)2\pi}$$

$$= -i\hbar A\left(-\frac{i}{\hbar}2\pi k\right)e^{-\frac{i}{\hbar}(kx-\omega t)2\pi}$$

$$= 2\pi k\left(Ae^{-i/\hbar(kx-\omega t)2\pi}\right) = 2\pi k|\psi\rangle$$

Since $k = \frac{p_x}{2\pi}$, $2\pi k = p_x$

In-Class Exercise: commutation relations of L_i

Demonstrate: $[L_i, L_j] = i\hbar L_k$

Choose $[L_x, L_y]$:

$$[L_x, L_y] = L_xL_y - L_yL_x = \left(-i\hbar\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(-i\hbar\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - \left(-i\hbar\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(-i\hbar\right)$$

$$= \left(i\hbar\right)^2\left[\overset{\mathrm{I}}{y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right)} - \overset{\mathrm{II}}{y\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial z}\right)} - \overset{\mathrm{III}}{z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right)} + \overset{\mathrm{IV}}{z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right)} - \left(\underset{\mathrm{V}}{z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z}\right)} - \underset{\mathrm{VI}}{z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}\right)} - \underset{\mathrm{VII}}{x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial z}\right)} + \underset{\mathrm{VIII}}{x\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial y}\right)}\right)\right]$$

I) $y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) = y\left[\frac{\partial}{\partial x}\frac{\partial z}{\partial z} + z\frac{\partial}{\partial z}\frac{\partial}{\partial x}\right] = y\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}\frac{\partial}{\partial x}$

II) $y\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial z}\right) = yx\frac{\partial}{\partial z}\frac{\partial}{\partial z} = yx\frac{\partial^2}{\partial z^2}$

III) $z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right) = z^2\frac{\partial}{\partial y}\frac{\partial}{\partial x}$

IV) $z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right) = zx\frac{\partial}{\partial y}\frac{\partial}{\partial z}$

V) $z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z}\right) = zy\frac{\partial}{\partial x}\frac{\partial}{\partial z}$

VI) $z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}\right) = z^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}$

VII) $x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial z}\right) = xy\frac{\partial^2}{\partial z^2}$

VIII) $x\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial y}\right) = x\left(\frac{\partial}{\partial y}\frac{\partial z}{\partial z} + z\frac{\partial}{\partial z}\frac{\partial}{\partial y}\right) = x\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}\frac{\partial}{\partial y}$

The formula can be reassembled:

$$[L_x, L_y] = (i\hbar)^2 \left[\mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV} - \mathrm{V} + \mathrm{VI} + \mathrm{VII} - \mathrm{VIII} \right]$$

note that $\mathrm{III} = \mathrm{VI}$ (cancel)

$\mathrm{II} = \mathrm{VII}$ (cancel)

the right half of $\mathrm{I} = \mathrm{V}$ (cancel)

the right half of $\mathrm{VIII} = \mathrm{IV}$ (cancel)

Leaving:

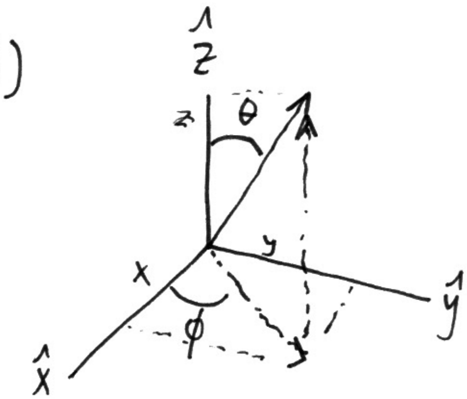
$$[L_x, L_y] = (i\hbar)^2 \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right] = (i\hbar)(i\hbar)\left(-\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\right)$$

$$= i\hbar\left[-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] = \boxed{i\hbar L_z}$$

Rinse and repeat for $[L_z, L_x]$ and $[L_y, L_z]$

In-Class Exercise: transformation of coordinates

Change $L_z(x,y,z) \to L_z(\rho,\theta,\phi)$



$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\theta = a\cos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\phi = a\tan(y/x)$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(2x)\left(x^2+y^2+z^2\right)^{-1/2}$$
$$= \frac{x}{\rho}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{1-\frac{z^2}{\rho^2}}\cdot\frac{\partial}{\partial x}\left(\frac{z}{\rho}\right)$$

$$= \frac{-z}{1-z^2/\rho^2}\cdot\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{-z}{1-z^2/\rho^2}\left(-\frac{1}{2}\frac{1}{\rho^3}\right)\frac{\partial}{\partial x}(x^2)$$

$$= \frac{-z\rho^2}{\rho^2-z^2}\left(-\frac{\cancel{1}}{\cancel{2}}\frac{1}{\rho^3}\right)\cancel{2}x = \frac{-zx}{\rho(x^2+y^2)}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right) = \frac{1}{1+y^2/x^2}\frac{\partial}{\partial x}(y/x) = \frac{y}{1+y^2/x^2}(-1)\frac{1}{x^2}$$

$$= -\frac{x^2y}{x^2+y^2}\frac{1}{x^2} = -\frac{y}{x^2+y^2}$$

So:

$$\frac{\partial f}{\partial x} = \left[\frac{x}{\rho}\frac{\partial f}{\partial \rho} + \left(\frac{-zx}{\rho(x^2+y^2)}\right)\frac{\partial f}{\partial \theta} + \left(-\frac{y}{x^2+y^2}\right)\frac{\partial f}{\partial \phi}\right]$$

similary:

$$\frac{\partial f}{\partial y} = \left[\frac{y}{\rho}\frac{\partial f}{\partial \rho} + \left(\frac{-yz}{\rho(x^2+y^2)}\right)\frac{\partial f}{\partial \theta} + \left(\frac{x}{x^2+y^2}\right)\frac{\partial f}{\partial \phi}\right]$$

Then:

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \left[\cancel{\frac{xy}{\rho}\frac{\partial}{\partial \rho}} - \cancel{\frac{xyz}{\rho(x^2+y^2)}\frac{\partial}{\partial \theta}} + \frac{x^2}{x^2+y^2}\frac{\partial}{\partial \phi}\right] - \left[\cancel{\frac{xy}{\rho}\frac{\partial}{\partial \rho}} - \cancel{\frac{xyz}{\rho(x^2+y^2)}\frac{\partial}{\partial \theta}} - \frac{y^2}{x^2+y^2}\frac{\partial}{\partial \phi}\right]$$

$$= \frac{x^2+y^2}{x^2+y^2}\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \phi}$$

Thus: $\boxed{L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial \phi}}$