Quantum Mechanics Core Proficiency Exam
January 2020

Problems provided by the Physics Faculty.
Exam assembled by the CPE Committee
(Balakishiyeva, Cooley, Olness (Chair), Sekula).

Instructions

The exam consists of two longer questions and two shorter questions. All four problems will be graded. You have two hours to work on the solutions. You are allowed one textbook of your choice, one math reference, and a calculator. Write your personal identifier (e.g. name, number, etc.) on the cover sheet of this exam. Show all work on scratch paper provided by the exam proctor. Staple your copy of the exam together with your scratch paper (exam on top). Hand that in when you are done.

Grading

Problem	Points Possible	Points Awarded			Final Score
1	10				
2	10				
3	20				
4	20				
Total	60				

Note for grader(s): if multiple people grade this exam, extra columns are provided under "Points Awarded" in the table above for different graders to report their scores.

Problem 1 [10 Points]

A particle of mass m constrained to move along the x-axis is described by a normalized solution of the time-dependent Schrödinger equation at time t given by the position-space wavefunction

$$\psi(x,t) = A \exp\left[-\lambda|x| - i\omega t\right]$$

where λ and ω are real parameters and A is a constant. Calculate the following in terms of m, λ and ω:

(a) (4 Points) The constant A

(b) (2 Points) The probability of observing the particle in the region $-1/\lambda < x < 0$

(c) (4 Points) The momentum-space wavefunction $\phi(p)$ where p is momentum

Problem 2 [10 Points]

Consider a quantum system with four linearly independent states. The Hamiltonian, in matrix form, is

$$H = V_0 \begin{bmatrix} 3-\varepsilon & 0 & 0 & 0 \\ 0 & 3 & -\varepsilon & 0 \\ 0 & -\varepsilon & 2 & 1 \\ 0 & 0 & 1 & 2+2\varepsilon \end{bmatrix}$$

The eigenvalue equation for the exact Hamiltonian leads to an equation which is difficult to solve. We will consider ε to be a small parameter and apply the perturbation method to find approximate values of possible energy measurements of this system. To start with, write the Hamiltonian in the $H^0 + H^1$ form. H^0 is zeroth-order in ε, while H^1 is a perturbation of H^0 and is first-order in ε.

(a) (4 Points) Find a set of orthonormal eigenstates of the unperturbed Hamiltonian H^0 which spans the Hilbert space; be sure to identify the eigenvalues.

(b) (6 Points) In part (a), you should have found one non-degenerate eigenstate of the unperturbed Hamiltonian H^0. Use perturbation theory to calculate the corrections to the energy of this non-degenerate state to first and second order in ε.

Problem 3 [20 Points]

(a) (1 point) For spin angular momentum operators, what is the commutator $[\hat{S}_x, \hat{S}_y]$?

A spin-$\frac{1}{2}$ particle is in the state $\psi = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. Here, $|+\rangle$ and $|-\rangle$ denote the eigenvectors of the operator S_z with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. What are:

(b) (1 point) $\langle \hat{S}_z \rangle$, the expectation value of the z-component of spin?

(c) (2 points) $\langle \hat{S}_x \rangle$, the expectation value of the x-component of spin?

(d) (2 points) $\langle \hat{S}_y \rangle$, the expectation value of the y-component of spin?

(e) (4 points) $\Delta S_x = \sigma_{S_x}$, the uncertainty in the x-component of spin?

(f) (4 points) $\Delta S_y = \sigma_{S_y}$, the uncertainty in the y-component of spin?

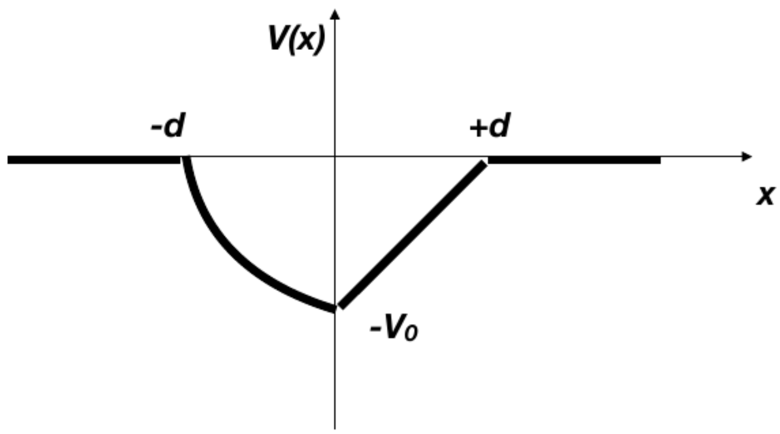
(g) (2 points) $\Delta S_x \Delta S_y$, the Heisenberg product of uncertainties?

(h) (4 points) Is the Uncertainty Principle violated? Explain.

Problem 4 [20 Points]

A particle with mass m interacts with following potential:

$$V(x) = \begin{cases} 0 & \text{for } |x| > d \\ V_0((x/d)^2 - 1) & \text{for } -d < x < 0 \\ V_0((x/d) - 1) & \text{for } 0 < x < +d \end{cases}$$



(a) (10 Points) Using the WKB approximation find the formula that will give the quantization of energy for bound states in this potential. If the formula is hard to represent in "E= ..." form, you may leave your answer in an intermediate-step expression, *but do not leave integrals.* Ideally, you should be able to show that:

$$\frac{(2mE_n + V_0)3\pi d}{8\sqrt{V_0}} + \sqrt{(2mE_n + V_0)}\frac{2d(2mE_n + V_0)}{V_0} = \left(n + \frac{1}{2}\right)\pi\hbar$$

(b) (5 Points) Assume that the potential is very deep ($|E|/V_0 \ll 1$) and find an approximate solution to the equation from part (a).

(c) (5 Points) From the answer in part (b) find the maximum possible number of bound states, n_{max}, assuming the potential is finite in depth.

You may find the following integrals useful in solving this problem:

$$\int \sqrt{1 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{1 - x^2} + \arcsin(x)\right) + C$$

$$\int \sqrt{1 - x}\, dx = -\frac{2}{3}(1 - x)^{\frac{3}{2}} + C$$

where in all cases C is a constant of integration.

Classical Mechanics Core Proficiency Exam
January 2020

Problems provided by the Physics Faculty.
Exam assembled by the CPE Committee
(Balakishiyeva, Cooley, Olness (Chair), Sekula).

Instructions

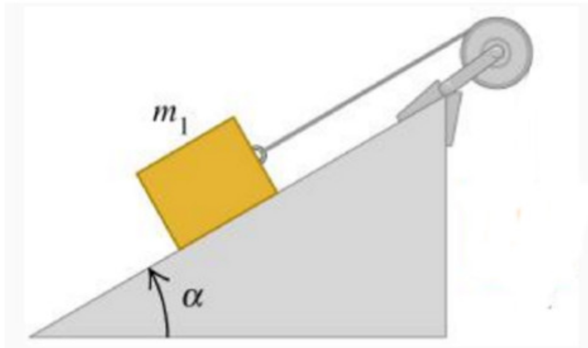
The exam consists of 6 questions, a mix of longer and shorter problems. All six problems will be graded. You have two hours to work on the solutions. This is a **closed-book exam**. Write your personal identifier (e.g. name, number, etc.) on the cover sheet of this exam. Show all work on scratch paper provided by the exam proctor. Staple your copy of the exam together with your scratch paper (exam on top). Hand that in when you are done.

Grading

Problem	Points Possible	Points Awarded			Final Score
1	10				
2	10				
3	10				
4	10				
5	20				
6	20				
Total	80				

Note for grader(s): if multiple people grade this exam, extra columns are provided under "Points Awarded" in the table above for different graders to report their scores.

Problem 1 [10 Points]



Consider an Atwood machine with mass m_1 on a frictionless ramp with a **MASSIVE** pulley of mass m_2 and moment of inertia of $I = (2/5)m_2r^2$. The string is wound around the pulley and does not slip. For coordinates, measure the distance of m_1 along the ramp to be x, and the rotation of the pulley to be θ.

(a) (5 Points) Compute the Lagrangian L, and obtain the associated equations of motion in terms of $\{x, x', \theta, \theta'\}$ using a Lagrange multiplier λ.

(b) (5 Points) Find the acceleration of the m_1. Also solve the Lagrange multiplier λ and compare this to the tension T in the string.

Problem 2 [10 Points]

"Skydiving" is the act of flying to an altitude of 4km above the surface of the Earth and jumping out of a plane. While eventually the skydiver uses a parachute to slow down for landing, much of the fall is spent falling under gravity with only air resistance ($F_{drag} = \frac{1}{2} C_d A \rho v^2$) to slow you.

Below an altitude of 4km, the average density of air is 1.016 kg/m^3. A human skydiver has a drag coefficient of about 1.0 and presents a cross-sectional area of about 0.80m^2 to the air as they fall. Loaded with equipment, the mass of the skydiver is about 100kg.

(a) [3 Points] What is the terminal velocity of the skydiver?

(b) [7 Points] At what altitude should the skydiver reach 99% of their terminal velocity?

Problem 3 [10 Points]

(a) (5 Points) A spider is hanging by a silk thread from a tree in Dallas. Find the orientation and the value of the equilibrium angle that the thread makes with the vertical (i.e. with the direction of gravity), taking into account the rotation of the Earth. Be sure to specify the direction of the deflection! (e.g. north, south, east, west, up, down, etc.) Assume that the latitude of Dallas is θ and we are in the northern hemisphere, and the radius of the Earth is R. [Important: note that the spider is stationary with respect to the surface of the Earth.]

(b) (5 Points) At latitude θ in the northern hemisphere, I shoot an arrow WEST with velocity v. What is the magnitude of the Coriolis force compared to the gravitational force? What is the direction of deflection due to the Coriolis force? (e.g. north, south, east, west, up, down, etc.)

Notes: $F_{\text{cor}} = 2m(v \times w)$, $F_{\text{cent}} = m(w \times r) \times w$

Problem 4 [10 Points]

(a) (5 Points) Kepler's laws of planetary motion are three scientific laws describing the motion of planets around the Sun. The second law states: "A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time." For an elliptical orbit, show this law is equivalent to conservation of angular momentum.

(b) (5 Points) Kepler's third law relates the period to the length of the semi-major axis of the orbit. For a circular orbit, compute the period T in terms of G, M, and r.

Problem 5 [20 Points]

Suppose you have a solid cylinder ($m = 0.5$ kg) attached to a horizontal spring ($k = 1.5$ N/m), and the cylinder rolls without slipping along a horizontal surface. If the cylinder is released from a point 15 cm from the system's equilibrium position, then:

(a) (10 Points) What are the translational and rotational kinetic energies of the cylinder when it is passing through its equilibrium position?

(b) (10 Points) What is the frequency of the simple harmonic motion due to the spring?

Problem 6 [20 Points]

(a) (2 Points) A muon has a lifetime of 2.2μs. The muon is observed by a person at rest on the surface of the Earth to be traveling at approximately the speed of light. How far (from the perspective of the Earth observer) will the muon typically travel before it decays? *For this question, ignore relativistic effects on the muon.*

(b) (3 Points) Now consider relativistic effects on the observed lifetime of the muon. To make a muon collider, let us suppose the muon must typically travel 10km. Compute the relativistic γ factor needed to achieve this goal. (That is, we want the muon to travel 10km during one lifetime of 2.2μs, observed from the perspective of a person at rest on the surface of the Earth, in the so-called "laboratory frame.")

(c) (1 Points) Use part b) to now compute β.

(d) (2 Points) An observer is at rest on an asteroid. From the perspective of that observer, two space ships approach each other each traveling at $v = 0.7c$. When they pass, what is their relative speed from the perspective of an observer on either ship? *(Hint, the answer is NOT* $1.4c$.*)*

(e) (6 Points) The Z boson is a neutral particle with a mass of 90 GeV/c^2. (For reference, in these units the proton mass is 1 GeV/c^2). If we create a Z boson in the process: $pp \to ppZ$, what is the minimum energy required of each initial-state proton if we collide them in the center-of-mass frame?

(f) (6 Points) Repeat part e), but now do it in the frame where one proton is at rest; what is the minimum energy of the initial-state moving proton required to create the Z boson?

For further reference:

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\
\beta &= \frac{v}{c} \\
\beta_{12} &= \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}
\end{aligned}
$$

Electricity and Magnetism Core Proficiency Exam
January 2020

Problems provided by the Physics Faculty.
Exam assembled by the CPE Committee
(Balakishiyeva, Cooley, Olness (Chair), Sekula).

Instructions

The exam consists of five questions, a mix of short and long problems. All five problems will be graded. You have two hours to work on the solutions. You are allowed a calculator. The exam is otherwise closed-book. Write your personal identifier (e.g. name, number, etc.) on the cover sheet of this exam. Show all work on scratch paper provided by the exam proctor. Staple your copy of the exam together with your scratch paper (exam on top). Hand that in when you are done.

Grading

Problem	Points Possible	Points Awarded			Final Score
1	10				
2	10				
3	10				
4	10				
5	20				
Total	60				

Note for grader(s): if multiple people grade this exam, extra columns are provided under "Points Awarded" in the table above for different graders to report their scores.

Problem 1 [10 Points]

Define in a sentence or two, with examples: refraction, diffraction, dispersion (*Pick 2 out of these 3 terms and define those 2 clearly using the guidelines to follow*) You should not use the word in its own definition; for example, do not use the verb “to refract” in the definition of “refraction”. (The “twitter limit” of 280 characters should be sufficient.) Each term you select is worth 5 Points.

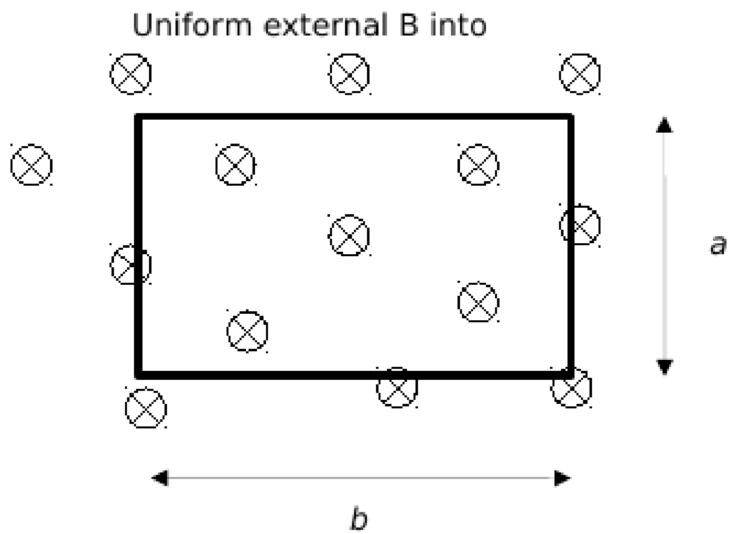
Problem 2 [10 Points]

A rectangular loop of wire with sides of length a and b sits in a uniform external magnetic field that is perpendicular to the plane of the loop and going into the page (see figure). The magnitude of the field is decaying linearly over time $B = B_0(1 - \alpha t)$.

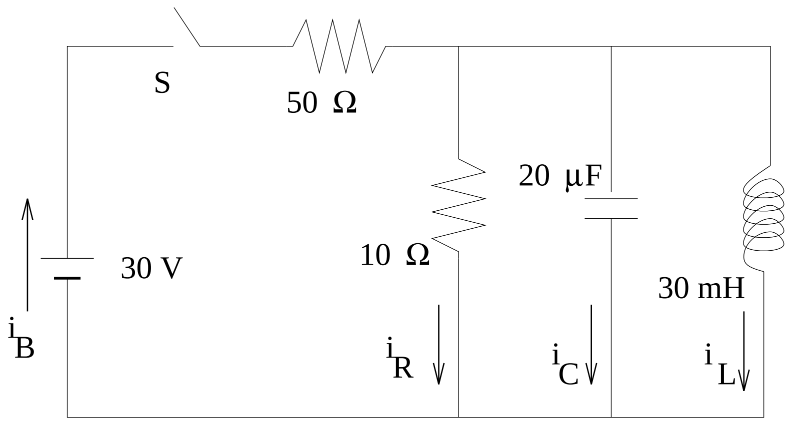
(a) (7 Points) If the resistance of the loop is R, find an expression for the current I induced in the loop.

(b) (2 Points) Does the current flow clockwise or anticlockwise (justify your answer)?

(c) (1 Point) Why can back-emf be neglected?



Problem 3 [10 Points]



(a) (3 points) What are the four currents an instant after the switch is closed? (Positve values if the currents are flowing in the directions indicated; negative if flowing the opposite direction.)

(b) (3 points) What are the four currents a long time after the switch has been closed?

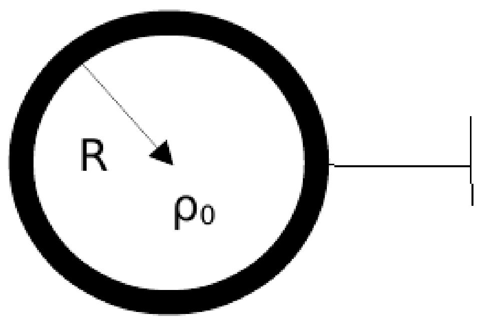
(c) (4 points) What are the four currents an instant after the switch is opened again, after having been closed for a long time?

Problem 4 [10 Points]

A perfectly conducting spherical shell of inner radius R and outer radius R' is filled with a uniform distribution of volume charge density:

$$\rho = \rho_0 \qquad (0 < r < R)$$

The conducting shell is grounded so that the scalar electric potential at radius $r = R$ is zero, $V(r = R) = 0$.



(a) (3 Points) What is the electric field vector E for $0 < r < R$?

(b) (4 Points) What is the scalar electric potential V for $0 < r < R$?

(c) (3 Points) What is the free surface charge density σ induced on the inside surface of the conducting shell at $r = R$?

Problem 5 [20 Points]

Derive the magnetic dipole moment of the spinning spherical shell of radius R carrying a uniform surface charge σ. The shell is spinning about the z-axis with angular velocity $\vec{\omega} = |\vec{\omega}|\hat{z}$. *Show as much work as possible.*

Statistical Mechanics Core Proficiency Exam
January 2020

Problems provided by the Physics Faculty.
Exam assembled by the CPE Committee
(Balakishiyeva, Cooley, Olness (Chair), Sekula).

Instructions

The exam consists of four problems. Three of them will be utilized to obtain your final score. Please try (and show as much work as possible for) all four problems. You have two hours to work on the solutions. You are allowed a statistical mechanics textbook (not a problem book) and a calculator. The exam is otherwise closed-book. Write your personal identifier (e.g. name, number, etc.) on the cover sheet of this exam. Show all work on scratch paper provided by the exam proctor. Staple your copy of the exam together with your scratch paper (exam on top). Hand that in when you are done.

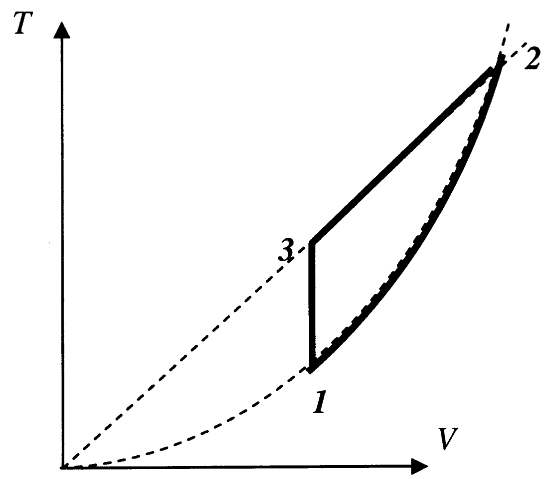
Grading

Problem	Points Possible	Points Awarded			Final Score
1	20				
2	20				
3	20				
4	20				
Total	60				

Note for grader(s): if multiple people grade this exam, extra columns are provided under "Points Awarded" in the table above for different graders to report their scores.

Problem 1 [20 Points]

One mole of helium follows cycle 1-2-3-1. During process 1-2, the temperature of the gas depends on its volume as $T = bV^2$, where b is a constant. During the cycle, the maximum temperature is four times the minimum temperature.



(a) (4 points) Redraw the cycle on a PV diagram, labeling the points 1, 2, and 3.

(b) (4 points) Express T_2 and T_3 in terms of T_1.

(c) (4 points) Express V_2 and V_3 in terms of V_1.

(d) (4 points) Express P_1, P_2, and P_3 in terms of T_1 and V_1.

(e) (4 points) What is the work done in a cycle in terms of T_1 and V_1?

Problem 2 [20 Points]

The enthalpy of a gaseous system (not an ideal gas!) is

$$H(p, S, N) = \frac{a}{8} S^2 N \sqrt{p}$$

where a is a constant, N is the number of particles, p is the pressure, and S is the entropy.

(a) (5 points) Derive an expression for the temperature T.

(b) (5 points) Derive an expression for the volume V.

(c) (10 points) What is the equation of state (an equation relating p, V, T, and N) for this gas?

Problem 3 [20 Points]

An ideal gas is composed of a mixture of N "red" atoms of mass m , N "blue" atoms of mass m, and N "green" atoms of mass m. Atoms of the same color are indistinguishable. Atoms of different color are distinguishable.

(a) (10 Points) Use the canonical ensemble to compute the entropy of this gas mixture.

(b) (10 Points) Compute the entropy of an ideal gas composed solely of $3N$ "red" atoms of mass m. Does it differ from that of the mixture described above. If so, by how much?

Problem 4 [20 Points]

Consider one mole of a gas that obeys the van der Waals equation of state. Its molar internal energy is given by

$$u = cT - \frac{a}{v}$$

where c is a new constant introduced for the above equation, T is the temperature of the gas, a is one of the two van der Waals coefficients (typically given as a and b), and v is the molar volume of the gas. Calculate the following:

(a) (10 Points) The molar heat capacity at fixed molar volume, c_v;

(b) (10 Points) The molar heat capacity at fixed pressure, c_p

The van der Waals equation of state is:

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT$$