Quantum Mechanics
May 2005

SHOW YOUR WORK IN ALL PROBLEMS

Work 4 of the 6 problems.

You may take the value of $\hbar$ to be 1 for all problems.

1. Consider a one-dimensional square well of width a, with 0 < x < a, with a single-particle states $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$, $E_n = (\frac{\pi^2 h^2}{2ma^2})n^2 \equiv E_1 n^2$, n = 1, 2, 3, ...

a) Suppose you put two identical spin-$\frac{1}{2}$ particles in this well, where the wavefunction is a product over space and spin parts, $\Psi = \psi(x_1, x_2)\chi(\overrightarrow{S_1}, \overrightarrow{S_2})$. Give the expression for the ground state wavefunction Ψ_1 and give its energy in terms of E_1. Do the same for the first excited state.

b) Suppose you put two identical spin-1 particles in the well, and the wavefunction is a product over space and spin parts, $\Psi = \psi(x_1, x_2)\chi(\overrightarrow{S_1}, \overrightarrow{S_2})$. If the spin part is a j = 2 state, $\chi(\overrightarrow{S_1}, \overrightarrow{S_2})$ = |j=2, m_j=0> = $\sqrt{\frac{1}{6}}$ (|1>|−1> + 2|0>|0> + |−1>|1>). (numbers in brackets are m_{s_1} and m_{s_2}, respectively. What is the associated lowest energy space wavefunction $\psi_0(x_1, x_2)$ and energy?

2) 1) The energy levels of a hydrogen atom are: $E_n = \frac{-13.6eV}{n^2}$. Consider the n = 2 to n = 1 (Lyman-α) transition. Find a formula to estimate the energy splitting in this line if the hydrogen is a mixture of normal hydrogen and heavy hydrogen (deuterium). Estimate numerically this splitting in electron volts.

3. The scattering wave function for a particular partial wave has the asymptotic form

$$\psi_l(r) \sim e^{-ikr} + \frac{(Rk)^2 - (10-3i)Rk + 4 - 40i}{(Rk)^2 - (10+3ik)Rk + 4 + 40i}\, e^{ikr}$$

where R is a constant with the units of length and k is the magnitude of the incoming momentum of the particle. The mass of the particle is m. What is the phase shift? Are there any bound states in the system with angular momentum l? If so, what are their energies? Are there any resonances with angular momentum l? If so, what are their energies and lifetimes?

4. The unperturbed Hamiltonian for a system of two identical bosons is given by

$$H_0 = \omega(a_1^\dagger a_1 + a_2^\dagger a_2)$$

where a and $a^\dagger$ are the usual creation and annihilation operators. The system is subjected to the perturbation

$$H_1 = \lambda(a_1^\dagger a_2^\dagger + a_1 a_2)$$

Find the time evolution operator in the interaction picture through order λ^2. If the system is originally in a state with unperturbed energy equal to 3ω, find the (interaction picture) wave function at time t through the same order in λ. At time $t = 0$ the operator q has the form

$$q(0) = (a_1 + a_1^\dagger)(a_2 + a_2^\dagger)$$

Find the expectation value of q as a function of time, also through second order in λ.

4. Calculate the Stark effect for each n = 2 state of a hypothetical hydrogen atom, which has no electron spin, in the following way:

a) Neglecting the effects of all n $\neq$ 2 states, set up the 4$\times$4 secular equation for H = $-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r} - e\epsilon z$ and obtain the first-order energy shifts for each n = 2 state due to the electric field. Give the first-order wave functions.

b) Derive an expression for the quadratic Stark effect for the 2p (m=1) state by using the second-order perturbation method. Express your results in terms of some radial integrals.

5. Consider a Helium atom modelled first with just the central potentials

$$V(r_1, r_2) = -2e^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

What is the degeneracy of the first excited state? Now consider the perturbation

$$\Delta V(r_1, r_2) = \frac{a l_1 \cdot l_2 + b s_1 \cdot s_2}{|\vec{r}_1 - \vec{r}_2|}$$

What is the first order shift in energy of each of the states discussed above? Is the degeneracy completely removed? You may wish to recall the relation

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{1}{2l+1} Y_l^{m*}(\theta_1, \phi_1) Y_l^m(\theta_2, \phi_2)$$

for Ilchenko, Dzyn, Rios,
Norton, Zhihuai Liang,
by Vega and Dalley
Part 1 of 2

Quantum Mechanics Qualifing Exam
Fall 2007

1. Consider the elastic scattering of an electrom of mass m, charge $-e$ and momentum $\hbar k$ from an atom of atomic number $Z << 100$. Ignoring spin effects calculate the angular distribution for electron scattering by the Coulomb field. The electrostatic potential which the incident electron sees is given by

$$\Phi(\mathbf{r}) = \frac{Ze}{r} - \int d^3r' \, \frac{e\, n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

where $n(\mathbf{r})$ is the electron density in the atom.

   (a) Show that the differential cross-section for elastic scattering in the Born approximation is given by,

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 Z^2 e^4}{|\mathbf{q}|^4} \left(1 - \frac{1}{Z}\tilde{n}(\mathbf{q})\right)^2$$

   where

$$\tilde{n}(\mathbf{q}) = \int d^3r \, e^{i\mathbf{q}\cdot\mathbf{r}} n(\mathbf{r})$$

   is the 'form factor' of the electronic charge distribution and $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer.

   (b) For a spherically symmetric atom show how small angle scattering can be used to determine the mean square radius of the charge distribution, $< r^2 >$, where,

$$< r^2 >= \int d^3r \, n(r) \, r^2.$$

   (c) Obtain an explicit result for scattering of an electron by a hydrogen atom in its ground state.

2. A particle with spin one-half (lambda hyperon) decays at rest ($l = 0$) into two particles with spin one-half (nucleon) and spin zero (pion).

   (a) Show that, in the representation in which the relative momentum of the decay products is diagonal, the final state wave functions corresponding to $m = \pm 1/2$ may, because of momentum conservation of the total angular momentum, be written in the form,

$$\psi_{\frac{1}{2},\frac{1}{2}}(p) = A_s \alpha + A_p(\cos\theta \, \alpha + e^{i\phi} \sin\theta \, \beta) \qquad (1)$$

$$\psi_{\frac{1}{2},-\frac{1}{2}}(p) = A_s \beta - A_p(\cos\theta \, \beta - e^{-i\phi} \sin\theta \, \alpha) \qquad (2)$$

   where α and β represent the spin 1/2 states of the nucleon spin-up and spin-down states respectively. Here θ is the polar angle of the momentum of the spin zero particle (the pion). Hint: The initial state is an eigenstate of the total angular momentum, thus the final state must also be an eigenstate of **J**. The final states are direct product states of the spin and orbital angular momentum states. The coordinate representation is used for the orbital angular momentum states (i.e. use the Spherical harmonics $Y_l^m(\theta, \phi)$). This problem is really about Clebsh-Gordon coefficients. Please neglect final state interactions.

(b) If we define

$$T_{\lambda\lambda'} = \psi_\lambda^* \psi_{\lambda'}$$

where, $\lambda = \pm 1/2$, then the 2×2 matrix T repesents the decay matrix which characterizes the angular distributions of the decay. Show that T can be written as:

$$T = I + 2Re(A_s^* A_p)\, \hat{p} \cdot \vec{\sigma}$$

where, $\hat{p}$ is the unit vector in the direction of the spin zero particle, I is the 2×2 identity matrix, and the σ's are the Pauli matrices.

(c) If the initial lambda hyperon is in the spin-up state then show that,

$$W = < \frac{1}{2}\frac{1}{2}|T|\frac{1}{2}\frac{1}{2} > = 1 + \lambda \cos\theta$$

where $\lambda = 2Re(A_s^* A_p)$.

(d) If the inital lambda is in a state of polarization **P** then show that,

$$W = 1 + \lambda \hat{p} \cdot \mathbf{P}$$

QM Fall 2007

for Ilchenko, Daye, Rios, Norton, Zhihao Liu, by Dalley and Vega
Part 2 of 2

The Harmonic Oscillator (HO) states in one dimension are governed by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}$$

where X and P are the position and momentum operators such that

$$X = \hbar^{1/2}\frac{(a + a^\dagger)}{(2m\omega)^{1/2}} \quad , \quad P = i\,\frac{(m\omega\hbar)^{1/2}}{2^{1/2}}(a^\dagger - a)\,,$$

and $a^\dagger$, a are the raising, lowering operators.
The orthonormal eigenvectors $|\,n>$, $n = 0,1,2,3,..$ satisfy

$$H\,|\,n> = \hbar\,\omega\,(n + \tfrac{1}{2})\,|\,n> \quad \text{and} \quad a\,|\,n> = n^{1/2}\,|\,n-1> .$$

A) On the same figure, sketch the potential, ground state eigenfunction and first excited eigenfunction for this problem. Specify one discrete and one continuous symmetry present in this problem and state their generators.

B) Use the $[X,P]$ commutator to find $[a, a^\dagger]$ and hence write H in terms of $a^\dagger a$.

C) Find the following matrix elements:
- **i)** $< n+1 \,|\, P \,|\, n >$
- **ii)** $< n \,|\, X^2 \,|\, n >$
- **iii)** $< n \,|\, X^3 \,|\, n >$

D) At time $t = 0$ a HO state is $|\,\psi(0)> = (\,|\,0> - \,|\,1>)/\sqrt{2}$. Find
- **i)** $< X(0) > = < \psi(0)\,|\,X\,|\,\psi(0) >$
- **ii)** $|\,\psi(t) >$
- **iii)** $< X(t) > = < \psi(t)\,|\,X\,|\,\psi(t) >$ as real function

E) If we model a quark of mass 5 MeV/c^2 in a nucleon as a particle in the ground state of a HO potential, roughly what value should we choose for ω if the quark is to contribute about 300 MeV/c^2 to the nucleon mass? Determine whether the r.m.s. size $\sqrt{<X^2>}$ is reasonable for a nucleon.

(a) Verify that the hydrogen atom has a ground-state wave function of the form $Ce^{-r/a}$, where $a = 4\pi\epsilon_0\hbar^2/(e^2 m)$.

What is the probability of finding the electron inside the nucleus of radius $R \ll a$?

[*You may assume that* $e^{-R/a} \approx 1$.]

(b) Write the angular-momentum operators L_1, L_2, L_3 in terms of $\mathbf{x}$ and $\mathbf{p}$ and verify the commutation relations

$$[L_1, L_2] = i\hbar L_3, \qquad [L_3, r^2] = 0, \qquad \text{where } r = |\mathbf{x}| .$$

Show that, for any function $f(r)$, the wave functions $\psi_0(\mathbf{x}) = f(r)$ and $\psi_1(\mathbf{x}) = x_3 f(r)$ are eigenfunctions of $\mathbf{L}^2 = L_1{}^2 + L_2{}^2 + L_3{}^2$ and L_3. What are the eigenvalues?

Show that suitable linear combinations of $\phi_1(\mathbf{x}) = (x_1^2 - x_2^2)f(r)$ and $\phi_2(\mathbf{x}) = x_1 x_2 f(r)$ are eigenfunctions of L_3 with eigenvalues $\pm 2\hbar$.

Quantum Mechanics Qualifying Exam
Feb 2008

You may use one textbook of your choice, one math reference, and a calculator.

The exam will last four hours.

Please choose two problems from Part I, two from Part II.

Part I

Choose two of three.

1. Consider a particle of mass m in a one-dimensional infinite square well potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a \\ \infty & x < 0, x > a \end{cases}$$

Let $|\phi_n>$ be the normalized eigenstate of the Hamiltonian of the system with eigenvalues E_n. The particle at time $t = 0$ is in the state

$$|\psi(0)>= a1|\phi_1> +a2|\phi_2> +a3|\phi_3> +a4|\phi_4>$$

1. Find E_n.
2. What is the probability, when the energy of the particle in the state $|\psi(0)>$ is measured, of finding a value smaller than $\frac{3\pi^2\hbar^2}{ma^2}$?
3. What is the mean value of the energy in the state $|\psi(0)>$?
4. Calculate the state vector $|\psi(t)>$ at time t.
5. What is the mean value of the energy in the state $|\psi(t)>$?
6. At some later time t_1, the energy is measured and the result is $\frac{8\pi^2\hbar^2}{ma^2}$. After the measurement, what is the state of the system?
7. If the energy is measured again at time $t_2 > t_1$, what values of the energy can be found and what are the probabilities?

2.

The state of a quantum mechanical particle confined to the x-axis is described by a wavefunction $\psi(x)$. The operator $\mathcal{P}$ is such that

$$\mathcal{P}\psi(x) = \psi(-x)$$

for all $\psi(x)$. Show that $\mathcal{P}$ has eigenvalues ± 1. Given that

$$\psi(x) = \psi_+(x) + \psi_-(x),$$

where $\psi_\pm(x)$ are eigenfunctions of $\mathcal{P}$ with eigenvalues ± 1, obtain expressions for $\psi_\pm(x)$ in terms of $\psi(x)$ and $\psi(-x)$. By considering the effect of the operator $\mathcal{P}$ on the function $x\psi(x)$, or otherwise, show that

$$\mathcal{P}x\mathcal{P}^{-1} = -x.$$

The inner product $\langle\psi_1, \psi_2\rangle$ of two wavefunctions ψ_1 and ψ_2 is defined by

$$\langle\psi_1, \psi_2\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx.$$

Let ψ be normalized such that $\langle\psi, \psi\rangle = 1$. Show that $\langle\mathcal{P}\psi, \psi\rangle = \langle\psi, \mathcal{P}\psi\rangle$ and that $\langle\psi, \mathcal{P}\psi\rangle$ is real. Obtain expressions for the inner products $\langle\psi_+, \psi_+\rangle$ and $\langle\psi_-, \psi_-\rangle$ in terms of $\langle\psi, \mathcal{P}\psi\rangle$. Hence show that

$$|\langle\psi, \mathcal{P}\psi\rangle| \leqslant 1.$$

Compute $\langle\psi_+, \psi_-\rangle$ and $\langle\psi_-, \psi_+\rangle$. For what wavefunctions ψ is it true that

$$|\langle\psi, \mathcal{P}\psi\rangle| = 1?$$

3. **(a)** Suppose that A is an observable, not explicitly dependent on time. Show that the expectation value of A in a stationary state does not depend on time.

A particle is in a state with wave function $\psi(x,t)$ which, at time $t=0$, has the form

$$\psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)) \, ,$$

where ψ_1 and ψ_2 are normalized eigenstates of the Hamiltonian with unequal energies E_1 and E_2. Show that the expectation value of A at time t is

$$\langle A \rangle = \frac{1}{2}\int_{-\infty}^{\infty} (\psi_1^* A\, \psi_1 + \psi_2^* A\, \psi_2)dx + \mathrm{Re}\left(e^{i(E_1-E_2)t/\hbar}\int_{-\infty}^{\infty} \psi_1^* A \psi_2 \, dx\right).$$

(b) The radial wave function R for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{e^2 R}{4\pi\epsilon_0 r} + \frac{\hbar^2 l(l+1)R}{2mr^2} = ER \, . \qquad (*)$$

Explain the origin of each of the terms in this equation. What are the allowed values of l?

The lowest-energy bound state solution of (*), for given l, takes the form $r^a e^{-br}$. Find a, b and the corresponding value of E, in terms of l.

A hydrogen atom makes a transition between two such states, corresponding to $l+1$ and l. What is the frequency of the photon emitted?

Part II

Choose two of three.

1. Calculate the elastic differential cross section for scattering off the potential given by,

$$V(r) = -V_0\, e^{-r/a} \qquad (V_0 > 0).$$

Use the Born approximation.

2. Show that the reduced matrix element for the angular momentum operator $\mathbf{J}$ is given by:

$$< j'||\mathbf{J}||j >= \hbar\sqrt{j(j+1)}$$

You might find one of these identities useful:

$$< j1j0|j1jj > = \sqrt{\frac{j}{(j+1)}} \tag{1}$$

$$< j2j0|j2jj > = \sqrt{\frac{j(2j+1)}{(j+1)(2j+3)}} \tag{2}$$

3. This problem involves the decay lifetime of hydrogen in an initial state with, $n = 3$, $l = 2$, and $m = 0$.

   (a) List the possible decay modes for this initial state.

   (b) Use the Dipole approximation to find the lifetime of this initial state.