# Qualifying Exam: Quantum Mechanics Fall, 2015

Name:________________________________

Instructions: There are two sections in this exam. Each section contains 3 problems. Choose 2 problems from each section. You should show all work. Indicate clearly which 2 problems in each section you wish to have graded. Attach your work to the back of this exam (e.g. by stapling).

# Section A
# Choose two of the following three problems

1. A proton is confined in an infinite square well of width 10fm. (The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by an infinite square well potential.)

   (a) Write the wave functions for the states $n = 1,\ n = 2$.

   (b) Calculate the energy and wavelength of the photon emitted when the proton undergoes a transition from the first excited state (n=2) to the ground state (n=1). In what region of the electromagnetic spectrum does this wavelength belong?

   (c) Determine the probability $P_n(1/a)$ that the particle is confined to the first $1/a$ of the width of the well. Comment on the n-dependence of $P_n(1/a)$ as $n \to \infty$.

2. Perform the following calculations:

   (a) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\Delta x$ for a quantum oscillator in its ground state.

   (b) Calculate $\langle p \rangle$, $\langle p^2 \rangle$, and $\Delta p$ for a quantum oscillator in its ground state.

   (c) Show that the ground state of the harmonic oscillator is an optimum state,

   $$\Delta x \Delta p = \hbar/2.$$

3. The operator $D = \frac{1}{2}\left(\underline{\mathbf{x}} \cdot \underline{\mathbf{p}} + \underline{\mathbf{p}} \cdot \underline{\mathbf{x}}\right)$ is associated with dilatations, i.e. rescalings of the position coordinates.

   (a) Evaluate the following commutators in terms of $x_j$ and $p_j$:

   $$[D, x_j] \text{ and } [D, p_j].$$

   (b) Show that for a free-particle Hamiltonian, $H_0$, that

   $$[D, H_0] = 2i\hbar H_0.$$

   (c) Consider the one-dimensional Hamiltonian

   $$H = \frac{p^2}{2m} + \frac{\lambda}{x^2}.$$

   Show that it obeys the same dilatation commutator relation as a free Hamiltonian.

# Section B

# Choose two of the following three problems

1. **NEUTRINO OSCILLATIONS.** It has been observed routinely since 1998 that neutrinos, nearly massless subatomic particles, "oscillate" or "mix." That is to say, if one begins with a pure beam of

electron-type neutrinos, $\nu_e$, after some time has passed it is possible to find muon-type neutrinos, $\nu_\mu$, in the beam, even though none were present at the beginning. Some of the $\nu_e$ are said to have "flavor oscillated," changing from electron-type to muon-type.

This behavior is possible if we can describe "flavor" (being an "electron" or a "muon", which we can label using $\nu_e$ and $\nu_\mu$) in one eigenbasis and "mass" (let us denote the mass eigenstates using the labels $\nu_1$ and $\nu_2$) in another eigenbasis. If these two eigenbases do not exactly coincide, it's possible to write the flavor eigenstates as linear combinations and thus for the flavor eigenstates to "mix" through the mass eigenstates.

(a) [2 Points] Let us write the transformation that related the two eigenbases as

$$|\nu_\alpha\rangle = \sum_{i=1}^{2} U_{\alpha i} |\nu_i\rangle$$

where $\alpha$ labels the flavor eigenstates and $i$ labels the mass eigenstates. *Show that the matrix, $U$, that relates the two eigenbases,*

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

*is unitary.* The angle, $\theta$, that parameterizes this matrix is called the "mixing angle" and describes the degree to which the flavor eigenstates transform into one another through the mass eigenstates.

(b) [4 Points] Since neutrinos are nearly massless, we have to be careful and incorporate some amount of special relativity into their eigenvectors. Assume that a free neutrino is propagating along the z-direction. We can write the eigenvector of the free neutrino mass eigenstate at any time, $t$, in terms of its eigenstate at $t = 0$, as

$$|\nu_i(t)\rangle = e^{-i(Et - p_i z)/\hbar} |\nu_i(0)\rangle,$$

where $i = 1, 2$ labels the mass eigenstates, $E(p_i)$ is the energy (momentum vector along the z-direction) of the $i^{th}$ mass eigenstate, and t ($z$) labels the time (distance traveled) since the creation of the $i^{th}$ mass eigenstate.

Neutrinos are nearly massless, and move at nearly the speed of light - this is the so-called "ultrarelativistic limit" where $pc \gg mc^2$ and thus $u \approx c$. Show that, in this limit at leading order, $Et - p_i z$ can be written as $m_i^2 c^3 z/(2E)$.

(c) [4 Points] A neutrino begins at $t = 0$, in a pure electron flavor eigenstate. *Show that the probability, $P$, that the electron neutrino of energy, $E$, is detected in the muon neutrino eigenstate, after traveling a distance, $z = L$, is given by*

$$P = \sin^2(2\theta) \sin^2\left(\frac{c^3 L(m_1^2 - m_2^2)}{4E\hbar}\right).$$

2. **SPIN AND SPIN AGAIN.** You are running an experiment in which a spin-1/2 particle is prepared in an eigenstate of the $S_z$ operator with quantum number $s_z(t = 0) = +\frac{1}{2}\hbar$. You then send the particle through a series of filters. The first filter passes only the x-component of the particle's spin, then the second filter only the y-component of the spin, and the last filter passes only z component of the spin.

(a) [2 Points] Thinking of spin as a classical angular momentum vector, what is the probability that you measure $s_z = -\frac{1}{2}\hbar$ after the last filter?

(b) [4 Points] Quantum mechanically, calculate the probability that you measure $s_z = -\frac{1}{2}\hbar$ after the last filter.

(c) [4 Points] Verify the Heisenberg Uncertainty Principle for the operators that measure the z- and x-components of the spin angular momentum:

$$\sigma_{S_z}\sigma_{S_x} \geq \frac{1}{2}|\langle[S_z, S_x]\rangle|.$$

3. **SCATTERING A SLOW NEUTRON.** A non-relativistic free neutron (of mass $m_n = 939.57\,\mathrm{MeV/c^2}$ and kinetic energy $K = 5.0\mathrm{MeV}$) traveling along a straight line in the z-direction then scatters off a heavy nuclear target. The target is at rest in the laboratory frame and has a mass of $2.22 \times 10^5\,\mathrm{MeV/c^2}$ and a radius of about $2.8 \times 10^{-15}\mathrm{m}$.

(a) [4 Points] Calculate the De Broglie wavelength of the neutron and show that we are safe to treat this scattering process in the low-energy/long-wavelength limit.

(b) [6 Points] The interaction potential responsible for this scattering process can be written as:

$$V(r) = -g^2\frac{e^{-\alpha m r}}{r},$$

where $g$ and $\alpha$ are constants, $m$ is a mass representing the mass of the particle mediating the scatter (e.g. an exchange of some particle between the neutron and the proton that causes the scatter) and $r$ is the distance from the center of the spherically-symmetric potential. *Working in the First Born Approximation, and for $g = 8.0 \times 10^{-7}\,\mathrm{MeV}^{\frac{1}{2}} \cdot \mathrm{m}^{\frac{1}{2}}$, $\alpha = 7.1 \times 10^{12}\frac{\mathrm{c^2}}{\mathrm{MeV \cdot m}}$, and $m = 139.57\,\mathrm{MeV/c^2}$, determine the total cross-section of this scattering process.*