by Dalley, Sokula, Kehoe

Quantum Mechanics Core Proficiency Exam
August 2017

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Instructions

The exam consists of two longer questions and two shorter questions. You are required to attempt to solve all four problems. You have two hours to work on the solutions. You are allowed one textbook of your choice, one math reference, and a calculator.

Problem 1

Consider the one-dimensional potential

$$V(x) = \frac{\lambda x^4}{4} + \frac{\lambda a x^3}{4} - \frac{\lambda a^2 x^2}{8}$$

(a) Find the points of classical equilibrium for a particle of mass m moving under the influence of this potential.

(b) Using the variational method, consider the trial wave function

$$\psi(x) = \left(\frac{\beta}{\pi}\right)^{4} e^{-\beta(x-x_0)^2/2}$$

where $x_0$is the global minimum found in (a). Evaluate the expectation value of the energy for this wave function and find the equation defining the optimal values of the parameter β, in order to get an estimate of the ground-state energy. Now take a special, but reasonable, value of the coupling constant, $\lambda = \hbar^2/(ma^6)$, and obtain the corresponding estimate of the ground-state energy.

(c) Write the potential in terms of the variable $x - x_0$ and, for small values of it, obtain the frequency ω of small oscillations around the global minimum.

Problem 2

A non-relativistic neutron (spin-$\frac{1}{2}$) is produced in the shielding surrounding an experiment. The neutron has a kinetic energy of $K = 1.0$MeV. The neutron then enters the experiment and interacts with an atomic nucleus (with a radius of $R = 4.72 \times 10^{-15}$m). Starting from this information, answer the following questions.

(a) The neutron is originally in a spin state given by $|n\rangle = \cos(\beta)|\chi^+\rangle + \sin(\beta)|\chi^-\rangle$, where $|\chi^+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents "spin up" and $|\chi^-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents "spin down". The nucleus with which the neutron interacts has one unpaired proton (also with spin-$\frac{1}{2}$), and is in the following initial state: $|N\rangle = |\chi^+\rangle$. Here, β is some angle representing the relative orientation of the neutron spin vector in space (e.g. relative to the spin axis of the nucleus). If the neutron is absorbed by the nucleus and forms a bound state with the nucleus, what is the probability that the neutron-nucleon system forms a singlet bound state? [HINT: treat the nucleus as a single particle, and ignore any radiation that would result from this process]

(b) Instead of forming a bound state, the neutron (mass m) might scatter off the nucleus (mass M). Note that $m = 939.57\text{MeV/c}^2$ and $M = 122.30 \times 10^3\text{MeV/c}^2$. Assume that the nucleus is originally at rest, and that the potential responsible for the scattering is given by

$$V(r) = -g^2 \frac{e^{-\alpha M r}}{r},$$

(where $g = 8.0 \times 10^{-7}\,\text{MeV}^{\frac{1}{2}} \cdot \text{m}^{\frac{1}{2}}$, $\alpha = 7.1 \times 10^{12}\,\frac{\text{c}^2}{\text{MeV}\cdot\text{m}}$, and r is the distance between the neutron and the nucleus). Show that the low-energy limit applies to this problem, and then, working in the first Born approximation, determine the total cross-section of this scattering process.

Problem 3

A hydrogen atom is in normalized state $\phi = a\psi_{100} + b\psi_{211}$ at time t $= 0$ and where a and b are nonzero.

(a) What is the expectation value of the Hamiltonian at time t=0?

(b) What is the expectation value of L_z at t=0?

(c) What is the expectation value of $(L_x^2 + L_y^2)$ at t=0?

(d) What is the expectation value of the Hamiltonian at time t=T?

(e) Is there a time when the wavefunction is purely ψ_{211}? If so, find the smallest time t>0 for which this is true. If not, explain your reasoning.

Problem 4

For each of the potential energy scenarios listed, provide the number of bound and scattering states. Potential answers are "0," "1 only," "1 or 2," "2 only," "finitely many," "denumerably infinite" (e.g. integer numbers), "nondenumerably infinite" (e.g. real numbers).

Potential energy	number of bound states	number of scattering states
Infinite square well		
Harmonic oscillator		
Finite square well		
Hydrogen atom		
Delta function well		
Delta function barrier		
Potential square step		
Free particle		