Work 4 of 5. We take top score.

Name: ____________________

Quantum Mechanics — Professors Gao/Olness

Qualifying Exam — **August 2006**

1) (20 points) a) Particles each with mass m and energy $E = \hbar^2/2ma^2$ are steadily flowing from the left ($x = -\infty$) towards a one-dimensional potential step: V(x) = 0 for x<0, and V(x) = $-3\hbar^2/2ma^2$ for x>0; where a is a constant of dimension length. Calculate the transmission coefficient T and the reflection coefficient R.

b) Consider a steady flow of particles with mass m and energy $E = \hbar^2/2ma^2$ towards a one-dimensional potential step: V(x) = 0 for x<0, and V(x) = $3\hbar^2/2ma^2$ for x>0; where a is a constant of dimension length. Determine the wave function u(x) (apart from normalization) in the regions $x < 0$ and $x > 0$ assuming that it is real and positive for $x > 0$. (You may put u(0) = 1).

2) (20 points) A particle (mass m, energy E and spin neglected) in a central potential $V(r)$ is in a stationary state given by the (time independent) wave function in spherical coordinates $\psi(r,\theta,\phi) = \frac{1}{8\sqrt{\pi}a^{5/2}} r e^{-r/2a} e^{i\phi} sin\theta$. Determine the orbital angular momentum quantum number and the magnetic quantum number. Determine energy E and $V(r)$ provided $V(r) \to 0$ as $r \to \infty$.

3) (20 points) A two-dimensional isotropic oscillator has the Hamiltonian $H = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{1}{2}m\omega^2(1 + bxy)(x^2 + y^2)$

The unharmonic term of strength b represents the deviation of the potential from harmonic form.

a) If b=0, determine the wavefunctions and energies of the three lowest energy states.

b) Now the unharmonic perturbation b≪1 is turned on. Determine the first order perturbation corrections to the energies of the three lowest states.

c) Evaluate the energy corrections in units of $\hbar\omega$ when $b = 0.1(\frac{\hbar}{m\omega})$.

SHOW YOUR WORK IN ALL PROBLEMS

4. The unperturbed Hamiltonian for a system of two independent oscillators is given by

$$H_0 = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2$$

where the a's and $a^\dagger$'s are the usual creation and annihilation operators. The system is subjected to the perturbation

$$H_1 = \lambda(a_1^\dagger a_1 a_2^\dagger + a_1^\dagger a_1 a_2)$$

Find the time evolution operator in the interaction picture through order λ^2. If the system is originally in the state $a_1^\dagger|0\rangle$, find the (interaction picture) wave function at time t through the same order in λ.

5. Assume that there are two flavors of neutrinos, ν_e and ν_μ, which are related to mass eigenstates ν_1 and ν_2 (masses m_1, m_2) by

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

where θ is a constant.

(a) For neutrinos of momentum p, where p/x is large compared to both m_1 and m_2, write down the time-dependent schrodinger equation describing neutrinos, in both the ($|\nu_1\rangle$, $|\nu_2\rangle$) and ($|\nu_e\rangle$, $|\nu_\mu\rangle$) bases.

(b) Now assume maximal mixing ($\theta = \pi/4$). If the neutrino starts as an electron neutrino (ν_e) at the origin and is observed at a distance L, find the probability that it is observed to be a muon neutrino (ν_μ).

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14. Consider an electron in a uniform magnetic field along the z direction. Let the result of a measurement be that the electron spin is along the positive y direction at t = 0. Find the Schrödinger state vector for the spin, and the average polarization (expectation value of S_x) along the x direction for t > 0.

20. A beam of spin $\frac{1}{2}$ atoms goes through three spin filters (Stern-Gerlach type measurements):

I. Allows only $S_z = +\hbar/2$ atoms to pass.

II. Allows only $S_n = +\hbar/2$ atoms to pass, where S_n is $\vec{S}\cdot\hat{n}$, where $\hat{n}$ makes an angle β with respect to the z axis in the xz plane.

III. Allows only $S_z = -\hbar/2$ atoms to pass.

If the wave function is normalized to unity after measurement I, what is the flux after measurement III? For what choice of β is this flux maximized?

For Pavel
sophia

Quantum Mechanics
January 2003

SHOW YOUR WORK IN ALL PROBLEMS

Work 3 of the 6 problems.

You may take the value of $\hbar$ to be 1 for all problems.

1. The Hamiltonian for a system with two states is:

$$H = \begin{pmatrix} 2 & 2+i \\ 2-i & -2 \end{pmatrix} \tag{1}$$

Initially the system is in the state:

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

An observable is represented (at $t = 0$) by the operator:

$$Q = \begin{pmatrix} 0 & 3-4i \\ 3+4i & 0 \end{pmatrix} \tag{3}$$

Find $Q(t)$ in the Heisenberg representation. Find the expectation value of the observable as a function of time.

2.The Hamiltonian for a three state system is:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & 1 & 0 \end{pmatrix} \tag{4}$$

Find the time evolution operator in the interaction representation through $\bigcirc(\lambda^2)$.

3. A simple harmonic oscillator is subject to the perturbation:

$$\Delta V = g e^{i(a-a^\dagger)}$$

Find the lowest order (order g) shift in the energy of the state of two quanta, that is, the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} a^\dagger a^\dagger |0\rangle$$

4. Find the first two eigenvalues and wave functions for the one dimensional (infinite) square well (of width L) with a repulsive delta function potential (of strength v) in its center. Draw pictures of the two wave functions.

5. Consider a system with a particle of orbital angular momentum $l_1 = 1$ and one of $l_2 = 2$. Use raising and/or lowering operators to find the (normalized) states of total angular momentum $l = 2$.

6. Compute the differential and total elastic scattering cross sections in the Born approximation for the Yukawa potential

$$U(r) = (A/r) exp{-\kappa r}.$$

Give the condition for the applicability of the approximation.

Quantum Mechanics
by McCartor/Scalise

Summer 2002

SHOW YOUR WORK IN ALL PROBLEMS

Work 4 of the following 5 problems.
You may take the value of $\hbar$ to be 1 for all problems.

1. For a 3-state system, the eigenvalues of the Hamiltonian are $E_1 = 1$, $E_2 = 2$ and $E_3 = 3$. The eigenstates corresponding to the first two eigenvalues are, respectively:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Another observable, Q, is represented by the operator:

$$Q = \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

At $t = 0$ the system is in the state $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If a large number of measurements of Q are made at time t, what will be the average of those measurements?

2. If the potential is given by

$$V = -2b^2 \text{sech}^2(br)$$

the s-wave Schrödinger equation can be solved exactly. The solution, for an arbitrary normalization, is given by

$$\psi = \frac{\sin(kr)}{kr} + \frac{bk \sinh(br)\cos(kr) - b^2 \cosh(br)\sin(kr)}{kr(k^2 + b^2)\cosh(br)}$$

Here, k is the particle's momentum. Assume that other partial waves are negligible so that this is the exact wave function. What is the scattering amplitude? What is the total cross section?

3. A non-relativistic particle of mass m and wave vector k moves in one dimension through a repulsive delta-function potential at the origin $V(x) = |a|\delta(x)$. Find the amplitudes of the reflected and transmitted waves.

4. A non-relativistic particle of mass m moves in one dimension. The eigenfunction for the ground state is

$$\psi(x) = \frac{A}{\cosh(cx)}$$

where A and c are constants. Assuming that the potential $V(x)$ vanishes at infinity, find the ground state energy.

5. Consider a non-relativistic electron bound in a hydrogen atom.

   (a) What is the expectation value of the energy in the first excited state?

   (b) What is the expectation value of the $(\text{energy})^2$ in the first excited state?

   (c) What is the uncertainty in the energy ΔE in the first excited state?

   (d) What is the uncertainty in the lifetime Δt of the first excited state?

   (e) How long should the electron stay in the first excited state?

   (f) Why is the experimentally measured lifetime of the first excited state considerably shorter?

SHOW YOUR WORK IN ALL PROBLEMS

Work 3 of the following 4 problems.
You may take the value of $\hbar$ to be 1 for all problems.

1. For a 2-state system the Hamiltonian is:

$$H = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

Another observable, Q, is represented by the operator:

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

At $t = 0$ the system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If Q is measured at time, t, what are the possible outcomes and what are their respective probabilities?

2. The unperturbed Hamiltonian for a harmonic operator is given by

$$H = \omega(a^\dagger a + \frac{1}{2})$$

The system is subjected to the perturbation

$$\lambda(a^\dagger a^\dagger + aa)$$

Find the time evolution operator in the interaction picture through order λ^2.

3. Find the S-wave phase shift for the potential

$$V(r) = \begin{cases} \infty, & \text{if } r \le a; \\ -V_0, & \text{if } a \le r \le b; \\ 0, & \text{otherwise.} \end{cases}$$

4. (a) Let $\hat{H}$ be the Hamiltonian operator of a physical system. Denote by $|\phi_n\rangle$ the eigenvectors of $\hat{H}$ with eigenvalues E_n. For an arbitrary operator $\hat{A}$, what is

$$\langle \phi_n | [\hat{A}, \hat{H}] | \phi_n \rangle$$

Simplify your result as much as possible.

(b) If the above result is true for all n, does $\hat{A}$ commute with $\hat{H}$? Explain in detail.

(c) The three Cartesian components of the angular momentum operator have non-vanishing commutators: $[\hat{L}_x, \hat{L}_y] \neq 0$, $[\hat{L}_y, \hat{L}_z] \neq 0$, $[\hat{L}_z, \hat{L}_x] \neq 0$, but it is still possible to find a state that is a simultaneous eigenfunction of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ (for example, the spherically symmetric S state with eigenvalue 0 for each component). Explain in detail why there is no contradiction here.

For Pavel
Sophia

Statistical Mechanics and Thermodynamics
January 2003

You may use Reif, lecture notes, and one book of math tables.

Choose one of the two following problems:

1. Consider a system composed of a very large number N of distinguishable atoms at rest and mutually noninteracting, each of which has only two (nondegenerate) energy levels: 0 and $\epsilon > 0$. Let E/N be the mean energy per atom in the limit $N \to \infty$.

   (a) What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium? What is the maximum attainable value or E/N if the system is in equilibrium (at positive temperature)?

   (b) For thermodynamic equilibrium compute the entropy per atom S/N as a function of E/N.

2. Consider a cylinder 1m long with a thin, massless piston clamped in such a way that it divides the cylinder into two equal parts. The cylinder is in a large heat bath at $T = 300°K$. The left side of the cylinder contains 1 mole of helium gas at 4 atm. The right contains helium gas at a pressure of 1 atm. Let the piston be released. (You may treat the helium as an ideal gas.)

   (a) What is the final equilibrium position?

   (b) How much heat will be transmitted to the bath in the process of equilibration? (Note that $R = 8.3$ J/mole °K, and 1 atm $= 1.013 \times 10^5$ N/m^2)

Choose one of the two following problems:

1. For a photon in mode i, the energy ϵ_i and angular frequency ω_i are related by $\epsilon_i = \hbar\omega_i$.

   (a) State or derive an expression for the partition function Z in terms of ω_i. From this, derive expressions for the average energy E, entropy S, and free energy F.

   (b) From these, derive an expression for the average pressure P, and show that the isothermal work done by the gas is

   $$dW = -\sum_i n_i \hbar \frac{d\omega_i}{dV} dV \tag{1}$$

   where n_i is the average number of photons in the ith mode.

   (c) Show that the radiation pressure is equal to one third of the energy density:

   $$P = \frac{1}{3}\frac{E}{V}\,. \tag{2}$$

For Du, Magan, Wang
by McCartor

MASTERS EXAM

QUANTUM MECHANICS

SPRING 00

You may take the value of $\hbar$ to be 1 for all problems.

WORK PROBLEM 1 AND *EITHER* PROBLEM 2 *OR* PROBLEM 3

1.The Hamiltonian for a 3-state system is:

$$H = \begin{pmatrix} 1 & 0 & \lambda i \\ 0 & 2 & \lambda \\ -\lambda i & \lambda & 1+\lambda^2 \end{pmatrix}$$

a) Treating λ as a small parameter, find the eigenvectors and eigenvalues through order λ^2.

b) Use these results to estimate the probability that, if the observable represented by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is measured to have the value 2 at $t = 0$, it will have the value 1 at time t.

c) Treating the Hamiltonian evaluated at $\lambda = 0$ as H_0 and the rest as V, find the time evolution operator in the interaction picture through order λ^2.

d) Use this result to estimate the probability in b).

2. A one-dimensional system has a potential consisting of three δ-functions:

$$V(x) = \epsilon(\delta(x-a) - \delta(x) + \delta(x+a))$$

If a beam of particles of momentum, k, is incident on this structure, what fraction of them will be reflected back ?

3. Two particles are in a one-dimensional box of length L. The system is in the first excited state. Find the probability density function for the separation of the particles — that is, the function $P(s)$ such that the probability of measuring $s \equiv |x_1 - x_2|$ to be between s and $s + ds$ will be given by $P(s)ds$ — under the assumption that: a) the particles are identical spin zero particles; b) the particles are identical spin $\frac{1}{2}$ particles in a state with total z-component of spin equal to 1.

QM Daeschler – Aug 00
by McCartor

Work three of the four problems.

You may take the value of $\hbar$ to be 1 for all problems.

1. For a 2-state system the Hamiltonian is:

$$H = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

At $t = 0$ the system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is the probability that at time t the system is in the state $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

2. The Hamiltonian for a 3-state system is:

$$H = \begin{pmatrix} 1+2\lambda^2 & 0 & \lambda i \\ 0 & 2 & \lambda \\ -\lambda i & \lambda & 1 \end{pmatrix}$$

a) Treating λ as a small parameter, find the eigenvectors and eigenvalues through order λ^2. b) Use these results to estimate the probability that, if the observable represented by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is measured to have the value 2 at $t = 0$, it will have the value 1 at time t. c) Treating the Hamiltonian evaluated at $\lambda = 0$ as H_0 and the rest as V, find the time evolution operator in the interaction picture through order λ^2. d) Use this result to estimate the probability in b).

3. A system consists of 2 electrons. In addition to the spin degrees of freedom the electrons can exist at either of 2 sites.

a) Specify an appropriate representation space for the system; what is its dimension? b) The Hamiltonian for the system is:

$$\epsilon \vec{S}_1 \cdot \vec{S}_2$$

if the electrons are at the same site, and:

$$\epsilon \frac{1}{2} \vec{S}_1 \cdot \vec{S}_2$$

if they are at different sites. What are the eigenvalues and eigenvectors of H? c) Find a Heisenberg representation of $(\vec{S}_1 + \vec{S}_2) \cdot \hat{x}$ (the x component of the sum of the spins).

4. For the potential

$$V(r) = -\frac{Ze^2}{r}e^{-\frac{r}{a}}$$

find, through second order in the Born approximation, the scattering amplitude for an incident particle of momentum k.

QUANTUM MECHANICS

MAY 99

SHOW YOUR WORK IN ALL PROBLEMS. CHOOSE TWO OF THESE THREE PROBLEMS.

1. Two identical, noninteracting, spin one-half particles of mass, M, are in the one dimensional box: $-L < x < L$. If the system is in a state which projects equally onto each of its ground states, and a measurement is made of the location of the two particles, what is the probability that one will be located at a negative value of x while the other is at a positive value? If, at the same time, a measurement is made of the z-components of the spins of the two particles, what is the probability that they will be found to have the same value?

   Now suppose the two particles interact through a potential, $\lambda \vec{s}_1 \cdot \vec{s}_2$. For what values of λ will your answers to the above problem change. To what will they change?

2. The Hamiltonian for a three-state system is:

$$H = H_0 + \lambda H_I$$

   Where:

$$H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

   and

$$H_I = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Work out the time evolution operator in the interaction picture through order λ^2.

3. A particle of mass m and charge q moves in a three-dimensional harmonic oscillator with spring constant k in all directions. A constant, uniform electric field of magnitude E_0 in the z direction is switched on for the interval $0 < t < \tau$, where τ is not especially large. Compute, to leading nontrivial order in perturbation theory and in any gauge desired, the probability that a particle starting in the ground state ends up in (a) an $n = 1$ state, and (b) an $n = 2$ state.