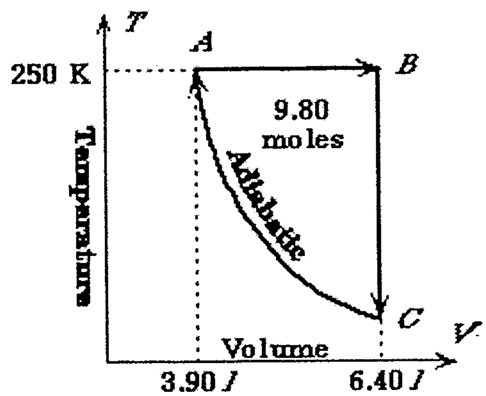
Statistical Mechanics and Thermodynamics
August 2011

You may refer to a statistical mechanics text of your choice, but no other notes or references.

Choose one of the following two problems:

1. An ideal gas is used as the work substance for a heat engine cycle shown in the Temperature-versus-Volume diagram. The gas is compressed adiabatically from 6.40 l (liters) at C to 3.90 l at A. The gas then expands isothermally at 250^0K while it does external work between A to B. The gas is then cooled at a constant volume between B and C. If there are 9.80 moles of ideal diatomic gas for which $c_v = (5/2)R$ and $c_p = (7/2)R$:



Before proceeding, state the relation between the change in energy and temperature for an ideal gas. State the relation between pressure and volume for adiabatic compression and expansion for an ideal gas. (You don't need to derive these. They may be useful for (a) to (e) below.)

(a) Sketch this cyclic process on a P-versus-V diagram, and determine the gas's pressure at the points A, B, and C. Also, determine the lowest temperature that the gas reaches.

(b) Calculate the heat flow during each part of the cycle, and the net heat absorbed by the engine.

(c) Calculate the external work done by the engine during the AB part of the cycle and the work done on the gas to compress it adiabatically from C back to its pressure and temperature at A. What is the net work done by the engine in each cycle?

(d) Calculate the thermal efficiency of this heat engine and the maximum possible efficiency that a heat engine could have that operated between the highest and lowest operating temperature of this engine.

(e) NEXT PAGE

(e) Finally, determine if this engine is a reversible engine by calculating the net change in entropy of the working gas throughout one cycle.

2. A metallic pipe inside the ATLAS detector of the Large Hadron Collider can be significantly deformed by the force of magnetic field existing inside the detector. The force moves atoms inside the pipe, so it also changes internal energy of the pipe.

   Explore the relation between the deformation of the pipe and its temperature using the following simplified model. For small deformations and temperature variations, the tension force F of the pipe is related to its length x and the absolute temperature T as

$$F = aT^2(x - x_0), \tag{1}$$

   where x_0 is the length in the unstretched state, and a is a positive constant. When $x = x_0$, the heat capacity C_x of the pipe (measured at constant length) is given by $C_x = bT$, where b is another positive constant.

   (a) Write down the fundamental thermodynamic relation between changes in the entropy, energy, and length of the system (ΔS, ΔE, and Δx).

   (b) Derive a Maxwell relation for $\left(\frac{\partial S}{\partial x}\right)_T$ (either from first principles or by analogy to one you know) that allows you to express this derivative in terms of x and T using the equation of state above.

   (c) Using T and x as free parameters and the result from (b), integrate relation (a) to find the entropy $S(T, x)$ at given T and x, if $S(T_0, x_0)$ is known. $x_2 > x_1$

   (d) If the pipe is stretched quasi-statically from length x_1 to length x_2 (with $\cancel{x_1 > x_2}$) while being thermally insulated, how will the pipe's temperature change? Explain your finding in a few sentences.

Choose two of the following three problems:

3. A simple harmonic one-dimensional oscillator has energy levels given by $E_n = (n + 1/2)\hbar\omega$, where $n = 0, 1, 2 \ldots$. Suppose that such an oscillator is in thermal contact with a heat reservoir at temperature T low enough that $kT/\hbar\omega \ll 1$.

   (a) Find the ratio of the probability of the oscillator being in the first excited state to the probability of its being in the ground state.

   (b) Assuming that only the ground state and first excited state are appreciably occupied, find the mean energy of the oscillator as a function of T.

   (c) Give the limiting values at low and high T and explain the result.

4. Graphite has a highly anisotropic crystalline layer structure. Assume as a simple model that each carbon atom in graphite can be regarded as performing simple harmonic oscillations in three dimensions. The restoring force in directions parallel to a layer are very large, and the natural frequencies of oscillations in the x and y directions lying within the plane of a layer are both equal to a value $\omega_{\|}$ which is so large that $\hbar\omega_{\|} \gg 300k$. But the restoring force perpendicular to a layer is small, and the frequency of oscillation $\omega_{\perp}$ of an atom in the z direction perpendicular to a layer is so small that $\hbar\omega_{\perp} \ll 300k$. On the basis of this model, give an expression for the molar specific heat (at constant volume) of graphite, and simplify it appropriately if $T = 300^0$K. Justify your simplification, and explain to what extent the equipartition theorem is applicable.

5. Treat an assembly of N_0 weakly interacting magnetic atoms per unit volume at a known temperature T *classically*. Their locations are fixed. In this case each magnetic moment $\boldsymbol{\mu}$ can have any arbitrary angle θ with respect to a given direction (call it the z direction). In the absence of a magnetic field, state the probability that the angle for a single atom lies between θ and $\theta + d\theta$. (To get the measure right, recall the definition of phase space in cartesian coordinates. Because the atoms are fixed, you may ignore momentum.) State the probability if the atom is now in the presence of a magnetic field of magnitude H in the z direction. Use this result to calculate the classical expression for the mean magnetic moment $\bar{M}_z$ for these N_0 atoms.