Statistical Mechanics and Thermodynamics
August 2012

You may refer to a statistical mechanics text of your choice, but no other notes or references.

Choose one of the following two problems:

1. (a) A quantity of water, initially at $10^o C$, is brought into contact with a heat reservoir at a temperature of $90^o C$. What is the entropy change of the entire system (water and reservoir) when the water reaches the temperature of the bath? Express the answer in terms of the heat capacity C of the water, and assume that C does not depend on temperature.

   (b) What is the entropy change of the entire system if we use two reservoirs to heat the water first from $10^o C$ to $50^o C$, and then from $50^o C$ to $90^o C$?

   (c) Can you suggest how, in principle, water might be heated from an initial temperature T_i to a final temperature T_f with no change in global entropy?

   (d) Explain why your calculation for the change in entropy for the water in the first part does not correspond to the actual physical situation, but does give the correct answer.

2. For an elastic rubber band of length L, at temperature T and under tension J, it is found experimentally that

$$g(T, L) \equiv \left(\frac{\partial J}{\partial T}\right)_L = \frac{aL}{L_0}\left[1 - \left(\frac{L_0}{L}\right)^3\right]$$

$$f(T, L) \equiv \left(\frac{\partial J}{\partial L}\right)_T = \frac{aT}{L_0}\left[1 + 2\left(\frac{L_0}{L}\right)^3\right]$$

   where L_0 is the length of the unstretched band (independent of temperature) and a is a constant.

   (a) Obtain the equation of state for this system. (That is, find a relation between J, T and L in the form of a function $J(T, L)$.

   (b) Relate changes in energy dE to dS and dL. Give a Maxwell relation relating a derivative involving entropy to $\left(\frac{\partial J}{\partial T}\right)_L$

   (c) Assume that the heat capacity at constant length of the band is a constant C_L. If the band is stretched, adiabatically and reversibly, from L_0 at an initial temperature T_i to a final length L_f, what is its final temperature T_f?

   (d) The band is now released, so that it contracts freely to its natural length L_0. If no heart is exchanged with its surroundings during this contraction, find the changes in its temperature and entropy.

Choose two of the following three problems:

3. A lattice in one dimension has N sites and is at temperature T. At each site there is an atom which can be in either of two energy states, $+\varepsilon$ or $-\varepsilon$. When L consecutive atoms are in the $+\varepsilon$ state, we say that they form a cluster of length L (provided that the atoms adjacent to the ends of the cluster are in the $-\varepsilon$ state). Consider such clusters in the limit $N \to \infty$, when the lattice boundaries can be ignored.

   (a) Which statistical ensemble describes the system? According to this ensemble, what are the probabilities P_+ and P_- for a site to have energies $+\varepsilon$ and $-\varepsilon$?

   (b) Compute the probability P_L that a given site belongs to a cluster of length L. (Define a site as being in a cluster of length zero if it has energy $-\varepsilon$.)

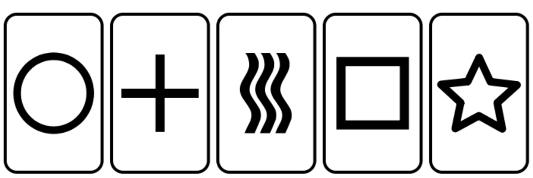
   (c) For a given site, calculate the mean length $< L >_T$ of the cluster it belongs to, and give its low- and high-temperature limits.

   The sum of a geometric series may be useful:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

   Recall that you can compute related sums by taking derivatives with respect to parameters such as x.

4. Zener cards are used to conduct experiments for extra-sensory perception (ESP), such as the ability of a human subject to guess a card without seeing it. There are five different Zener card faces: a circle (one line), a Greek cross (two lines), wavy lines (three lines), a square (four lines), and a star (which counts as five lines). There are 25 cards in a deck, five of each design.



   In each guessing trial, a card is randomly drawn from the deck, examined, and put back into the deck. The deck is reshuffled, and the procedure repeats.

   (a) In a sequence of 25 trials, what is the probability that a subject (without ESP) will correctly guess the card in every trial?

   (b) What is the probability that this subject will correctly guess at least one card (that is, one or more) in 25 trials?

(c) What is the probability for making n correct guesses in 25 trials, with n between 0 and 25? That is, give the probability distribution for the number of correct guesses as a function of n and total number of cards $N = 25$. (A useful check is that it is normalized to 1.)

(d) Which n is most likely (that is, the mode, or maximum for the distribution)? What is the mean n?

(e) The cards are dumped into a heat bath at absolute temperature T. Assume the cards can only be face up or face down, and that because of heavy ink, the energy of each card is higher when face up by an amount E_0 times the number of lines (see above). Give the partition function for the deck.

(f) What is the average number of lines visible (face up) at temperature T?

5. Find an expression for the partition function $Z(T, L)$ for a single quantum-mechanical particle in an infinite one-dimensional square well of width L. Simplify it in the limit of low temperature. Obtain the heat capacity c_L and an equation of state relating pressure P, T and L. (Keep enough terms to obtain a nontrivial answer for c_L.)

   Recall that the energy levels for this system are given by

$$E(n) = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \quad n = 1, 2, 3, \ldots.$$