Statistical Mechanics and Thermodynamics
August 2014

You may refer to a statistical mechanics text of your choice, but no other notes or references.

1. Consider a system A with a single particle which can be in one of only two states, the ground state with energy 0, and an excited state with energy ϵ. The system B has three of the same type of particle, each of which can be in one of these two states. Assume all the particles are distinguishable.

   (a) Initially the two systems are isolated and separated, with the particle in A in the ground state, and in B two are in the ground state and one is in the excited state. They are then brought into thermal contact, but the combined system $A+B$ remains isolated. After the combined system reaches equilibrium, compute the probability P_0 that the particle in A is in the ground state.

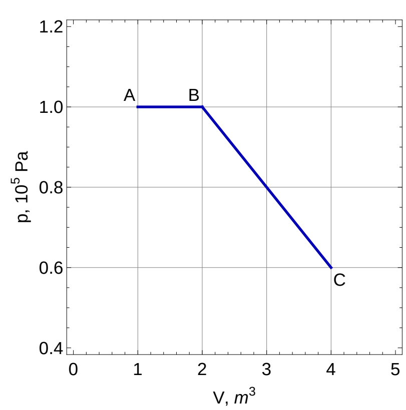
   (b) Repeat for the case where in the initially separated systems, two of the particles in B are excited and one is in the ground state, and the particle in A is in the ground state.

Answer one of the following two problems.

2. The entropy S of a certain rarified gas in a reservoir was found to satisfy the following relation:

$$S = N\,k \ln\left[\frac{V}{N}\left(\frac{E}{kN}\right)^{7/2}\right], \tag{1}$$

where E is the internal energy, V is the volume, N is the particle number, and k is the Boltzmann's constant.



   (a) Find an expression for the energy E as a function of temperature T.

   (b) Derive an equation of state for the gas relating pressure p to T, N and V.

   (c) How much energy is needed to heat up 2 moles of the gas by 20 K at constant pressure?

   (d) What is the work done by the gas in the process $A \to B \to C$ in the figure?

   (e) How much heat is absorbed in this process?

   (f) Can the gas described by Eq. (1) be ideal? Can it be monatomic? Can Eq. (1) be an accurate description at low T? Explain.

3. Consider a paramagnetic substance whose equation of state is given by Curie's Law

$$M = \frac{DH}{T}$$

where T is the absolute temperature, M is the magnetization, H is the magnetic field, and D is a material-specific constant. The internal energy is given by

$$E = CT$$

where C is a constant. Under certain circumstances, you may define the work done *by* this system to be $(-)HdM$.

(a) State the fundamental thermodynamic relation between the quantities above and the entropy S. (Volume is fixed and may be ignored.)

(b) Show that on an adiabatic curve, H and M are related by

$$\frac{H}{H_0} = \frac{M}{M_0} \exp\left[\left(M^2 - M_0^2\right) / (2CD)\right]$$

where H_0 and M_0 are values at a reference point, and T and M satisfy

$$\frac{T}{T_0} = \exp\left[\left(M^2 - M_0^2\right) / (2CD)\right]$$

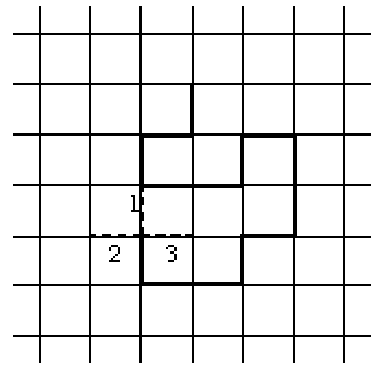
(c) Sketch the closed curve on the $H - M$ plane for a Carnot cycle, made of two adiabatic curves intersecting two isothermal curves. Be careful to draw them with the appropriate shape. Label the points of intersection from 1 to 4, and indicate the direction of the cycle.

(d) Compute the heat absorbed during each of the four parts of the cycle in terms of the values $M_1, M_2, \ldots$ at the four intersection points and the two different values for T on the isotherms. (It will help to recall that, for this system, $E = CT$.)

(e) Give an appropriate definition for the efficiency η and give its value.

Answer one of the following two problems.

4. A simple model for a polymer in two dimensions is that of a path on a square lattice. At every lattice point the polymer can either go straight (option 1 in the figure) or choose between the two directions at a 90^o angle with respect to its current direction (options 2 and 3 in the figure.) Each time it bends in a 90^o angle, it pays a bending energy ϵ. Thus, for a given "shape" of the polymer the total bending energy of the polymer is ϵ times the number of 90^o turns. We assume that the starting segment of the polymer is fixed somewhere on the lattice and that the polymer consists of $N+1$ segments. Each possible shape of the polymer is a state of this system.



(a) Calculate the partition function of this polymer as a function of temperature T and the number of joints N. (If you have trouble simplifying it, you might consider the cases where $N = 1$ and 2.)

(b) Calculate the average energy E and heat capacity C as a function of T and N.

(c) Give an expression for the average number of 90^o bends B as a function of T and N.

(d) How does B behave in the limits of large and small T? Explain.

5. A polarized target for a particle scattering experiment is a cylinder of length L and radius R filled with a diffuse gas of N_0 atoms with spin 1/2 and magnetic moment along the z axis of $\pm\mu$. The target is placed inside a solenoid that creates a magnetic field $\vec{H}$ pointing along the z direction. Because of minor imperfections, H changes as a function of the distance r from the axis of the cylinder, $H_z(r) = h_0 + h_1 r$. Find the average magnetic moment M_z of the target at temperature T.