Statistical Mechanics Core Proficiency Exam, January 20, 2018

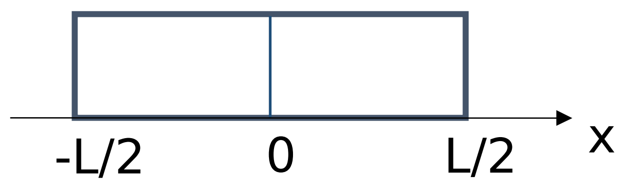
You can use a textbook of your choice and no other notes

Short problems: solve 2 out of 3

S1. You have two urns and a small bag. One urn contains 100 balls, numbered 1 to 100. The other urn is empty. 100 small slips of paper with numbers 1 to 100 written in sequence are put into the bag. Each second, a paper slip is drawn randomly from the bag, the number is noted, and the slip is returned to the bag. The ball bearing that number is moved from its current urn to the other urn. After a while, we would expect that the system settles into an equilibrium state in which there are about 50 balls in each urn. The ball counts in the urns will fluctuate about the 50-50 distribution, however, it would appear highly unlikely that the system would ever return to the state in which all 100 balls are back in the first urn. Nevertheless, the Poincare' Recurrence Theorem that such situation will surely reoccur. But after how long?

1) Estimate the likely difference of the numbers of balls in two urns that may result because of a fluctuation from the 50-50 equilibrium.
2) What is the Poincare' recurrence time in years for the urns to return to the state with 100 balls in one and zero in the other? Explain.
3) What is the Poincare' recurrence time for one liter of helium at STP to be found in half of its container? Explain.

S2. An insulated container of mass M and length L is at rest on a horizontal frictionless surface. The container is filled with an ideal gas of unknown mass m and temperature T. The container is divided in half by a light movable insulated piston.



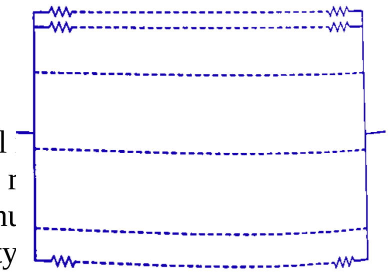
A heater is turned on inside the left part of the container, bringing the temperature of gas there to $2T$. The temperature of the gas inside the right part of the container remains unchanged. As a result, the container moves a distance x along the surface. Find the mass m of the gas.

S3. The molecules of an imaginary ideal gas have internal energy that are equally spaced so that the n-th energy eigenvalue is $E_n = n\varepsilon$, where $n = 0,1,2\ldots$. The degeneracy of the n-th energy level is $n+1$. Calculate the contribution to the thermal energy of the internal energy states. **Hint:** $\sum_{m=0}^{\infty} x^m = 1/(1-x)$ for $x = e^{-\beta\varepsilon}$.

Long problems: solve 2 out of 3

L1. a) Prove that, if $y(x)$ is a smooth function of a random variable x with average $\acute{x}$ and a small uncertainty Δx (so that $\Delta x / |\acute{x}| \ll 1$), then

$\acute{y} \approx y(\acute{x})$ and $\Delta y \approx |y'(\acute{x})| \Delta x$.



b) A rectangular array of resistors shown in the figure has K parallel resistors in series. The resistors are drawn from a lot with an average r and an uncertainty of 10%. Assuming that K and L are large nu the relations from part a), calculate the average of, and the uncertainty resistance of the array.

L2. A system is composed of K one-dimensional classical oscillators. Assume that the potential for the oscillators contains a small quartic "anharmonic term":

$$V(x) = \frac{k_0}{2} x^2 + \alpha x^4,$$

where $\alpha \langle x^4 \rangle \ll kT$. To the first order in the parameter α , derive the anharmonic correction to the Dulong—Petit law. **Hint:** You may need to find an approximate expression for the partition function of a single oscillator by expanding the $\exp(-\alpha x^4)$ term as the Taylor series.

L3. A system is composed of a large number N of one-dimensional quantum harmonic oscillators whose angular frequencies are distributed over the range $\omega_a \leq \omega \leq \omega_b$ with a frequency distribution function $D(\omega) = A/\omega$.

a) What is the value of A , given that there are N oscillators?
b) What is the partition function of a single harmonic oscillator?
c) Find the energy E of N oscillators in the thermodynamic equilibrium at temperature T .
d) Calculate the specific heat C per oscillator.
e) Evaluate the specific heat in the limits of high and low T .