Pulse Transmission and Cable Properties

GOAL

- To understand how voltage and current pulses are transmitted along a transmission cable.
- How to terminate transmission cables properly.

Introduction

In your introductory electricity and magnetism classes you typically assumed that the circuits you studied were small enough so that the instantaneous values of current and voltage along a conductor were independent of position. In many experimental situations this assumption breaks down and the current and voltage will depend on position along the conductor even in the absence of discrete circuit elements. Additionally, you will often need in experimental physics to transmit a voltage or charge pulse from one device to another with as little distortion of that pulse shape as possible. Doing so is not necessarily a trivial matter and becomes increasingly difficult as the pulse duration shortens. At all times the transmission of the pulse is governed by Maxwell's equations, which implies that appropriate boundary conditions must be satisfied when the conduction path, usually a cable, interfaces with either a source or a load. You will see below that the most important macroscopic characteristic of a cable is its "characteristic impedance." Failure to "terminate" a cable with its characteristic impedance will lead to reflections of the traveling pulse that distort its shape and almost always degrades its usefulness. After today's lab, you will appreciate that understanding a "simple" cable isn't completely trivial.

What follows is especially relevant when the width or risetime of a voltage pulse is comparable to the round trip transit time for a lightwave travelling along the cable. For most of the signals you look at on an oscilloscope in this course, this condition is *not* satisfied and so you don't need to care about the material in this lab. However, when you deal with "fast" signals, those that do satisfy the round trip condition, you will need to be concerned about the issues discussed in this lab.

Theory of the Transmission Line/Coaxial Cable

In the laboratory you almost always transmit voltages and currents from one place to another using a cable. For our purposes, a cable is a pair of parallel conductors, each of constant cross-section, although not necessarily identical. The conductors need not be side by side but one can be inside the other, like coaxial cables, the most common cable type you will use in the lab to interconnect two electronic devices. The inner conductor of a coaxial cable is typically a solid copper wire and the outer conductor is a larger diameter braided cylinder. The volume between the two coaxial conductors is filled with a dielectric. When the cable is conducting a signal, the central wire is the conductor that is at some potential with respect to ground while the outer conductor is usually at 0 volts potential, i.e., it is "grounded." The outside conductor is grounded to minimize the likelihood that transient voltages will be induced on the central conductor by any ambient electric fields in the region where the cable is used. The outer conductor is referred to as the "shield."

Although it is often said that the central conductor of the coaxial cable "carries" the signal, this is not strictly correct. The coaxial cable functions as a wave guide and the electric and magnetic fields that comprise the electromagnetic pulse travel through the volume occupied by the dielectric. Since we are concerned with time varying voltages, the mutual capacitance and inductance of our two-conductor/dielectric "system" will have to be taken into account. Additionally, since copper has a finite resistance (alas, copper isn't even a superconductor), we should also consider the finite resistance of the coaxial cable. The implications of these statements will be explained in detail presently.

Refer to figure 1 for a schematic arrangement of a cable of length l connecting a source and some load of impedance Z_L. The position along the cable length is denoted by x. The source at $x = 0$ is assumed to produce a voltage $\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$ and to have an output impedance Z_{out}. **We want to find the voltage and current as functions of position along the cable.** The discussion that follows does not depend on the assumption that the cable is coaxial.

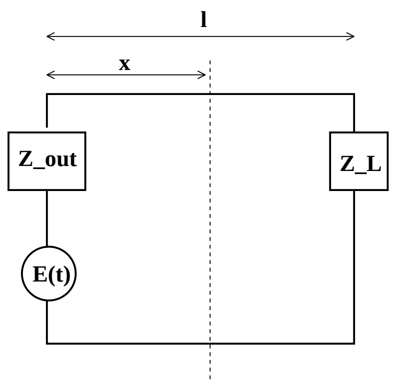

Figure 1: Schematic representation of a source with output impedance Z_{out} connected by a transmission line to a load of impedance Z_L.

Since the cable has a finite length, we will not make the assumption that cables properties like resistance, capacitance and inductance are all located at a single point. (What point would we use?) Instead, we will treat these parameters as being distributed along the cable between its two ends. Let R' be the cable resistance per unit length and L' be the cable inductance per unit length. The series impedance of an infinitesimal length dx of cable is then $(R' + j\omega L')\,dx$. (We will use the standard convention $j = \sqrt{-1}$ so as to reserve the symbol i for electric current.) Since our cable will contain more than one conductor, there will also be a capacitance C' per unit length. Since it may well be the case that the insulation between the two conductors is imperfect, there can be a leakage current between them that can be described by a *conductance* G' per unit length. (You can think of conductance as the reciprocal of resistance.) We therefore model a differential length of our cable as shown in figure 2. Again, no assumption is made that the cable is coaxial.

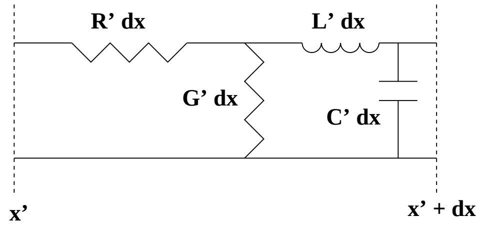

Figure 2: Schematic representation of an infinitesimal portion of a transmission line.

We can apply Kirchoff's law to the infinitesimal portion of the cable. If $I(x)$ is the current at x then the current at $I(x + dx)$ is less than this by the amount $YV(x)$ that leaks across the cable, where $Y(x)$ is the admittance[1] between the two conductors and $V(x)$ is the voltage difference between the two conductors. Hence, $I(x + dx) = I(x) - YV(x) = I(x) - (G' + j\omega C')V\,dx = I(x) + (\partial I/\partial x)\,dx$, so that

$$\frac{\partial I}{\partial x} = -(G' + j\omega C')V. \tag{1}$$

By applying Kirchoff's law to the path bounded by the dashed lines in figure 2, we find that $-(R' + j\omega L')I\,dx - V(x + dx) + V(x) = 0$, which implies that

$$\frac{\partial V}{\partial x} = -(R' + j\omega L')I. \tag{2}$$

Differentiating (1) and using (2) yields an equation that involves only V

$$\frac{\partial^2 V}{\partial x^2} - \gamma^2 V = 0, \tag{3}$$

where $\gamma = \alpha + j\beta = [(R' + j\omega L')(G' + j\omega C')]^{1/2}$ and is called the *propagation constant* (even though it depends on frequency). The real part of γ is called the "attenuation constant" and whose significance will be clearer shortly. The imaginary part β is the "phase constant." Additionally, we can differentiate (1) and use (2) to find a similar equation for I

$$\frac{\partial^2 I}{\partial x^2} - \gamma^2 I = 0. \tag{4}$$

[1]Admittance is the reciprocal of impedance.

Notice that I and V satisfy the same differential equation. We therefore expect them to have solutions of a similar form.

The solution to (3) is

$$V(x) = V_1 e^{\gamma x} + V_2 e^{-\gamma x}, \tag{5}$$

where V_1 and V_2 are integration constants. The solution $V(x,t)$ is found by multiplying (5) by the harmonic dependence $e^{j\omega t}$ to yield

$$\begin{aligned} V(x,t) &= V_1\, e^{\alpha x} e^{j(\omega t+\beta x)} + V_2\, e^{-\alpha x} e^{j(\omega t-\beta x)} && (6)\\ &\equiv V_r(x,t) + V_i(x,t). && (7)\end{aligned}$$

The term containing $\omega t + \beta x$ represents a *reflected* wave $V_r(x,t)$, reflected from the load and traveling in the negative x-direction along the transmission line. The factor $e^{\alpha x}$ indicates that this wave *diminishes* in magnitude as it propagates in the negative x-direction. (Now you can see why α is called the attenuation constant.) The term containing $\omega t - \beta x$ represents the incident wave $V_i(x,t)$, incident on the load and traveling in the positive x-direction. The factor $e^{-\alpha x}$ indicates that this wave diminishes in magnitude as it travels in the positive x-direction. The voltage at any point along the transmission line is a *superposition* of these two traveling waves. It is easy to show that the general form of the solution for $I(x,t)$ is similar to that for V and also contains exponentially damped waves propagating in opposite directions

$$\begin{aligned} I(x,t) &= -\frac{V_1}{(Z/Y)^{1/2}}\, e^{\alpha x} e^{j(\omega t+\beta x)} + \frac{V_2}{(Z/Y)^{1/2}} e^{-\alpha x}\, e^{j(\omega t-\beta x)} && (8)\\ &\equiv I_i(x,t) + I_r(x,t). && (9)\end{aligned}$$

For the waves described by (7) and (9), β equals $2\pi/\lambda$, where λ is the wavelength, and ω/β is the wave speed. If R' and G' are small enough or if the frequency ω is large enough so that $\omega L' \gg R'$ and $\omega C' \gg G$, then $\beta = \gamma \simeq \omega\sqrt{L'C'}$. This implies then that the wave speed v is given by

$$v = \frac{\omega}{\beta} \simeq \frac{1}{\sqrt{L'C'}}. \tag{10}$$

If we consider just the wave traveling in the positive x-direction, then we will see shortly that the ratio of the voltage $V_i(x,t)$ between the two conductors to the current flowing through the conductors $i_i(x,t)$ is an important parameter of the cable. This ratio has the units of ohms and is termed the **characteristic impedance** Z_0 of the transmission line.

$$Z_0 = \frac{V_i(x,t)}{i_i(x,t)} = \sqrt{\frac{Z}{Y}} \tag{11}$$

$$= \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}. \tag{12}$$

For the case of an ideal transmission line where the resistance is negligible ($R' \simeq 0$) and the dielectric is assumed perfectly insulating ($G \simeq 0$),

$$Z_0 = \sqrt{\frac{L'}{C'}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln \frac{b}{a}. \tag{13}$$

Here, we have used the explicit expressions for L' and C' appropriate for a coaxial cable[2]

$$L' = \frac{\mu}{2\pi} \ln \frac{b}{a} \tag{14}$$

$$C' = \frac{2\pi\epsilon}{\ln(b/a)}. \tag{15}$$

The permittivity of the dielectric is denoted by ϵ and the permeability is denoted by μ. Typically, for coaxial cables you will use, $\epsilon \simeq 3\epsilon_0$ and $\mu \simeq \mu_0$. (The dielectric is usually polyethylene or Teflon.) The inner conductor radius is a and b is the outer conductor radius.

Reflection Coefficient from the Load

What is the significance of Z_0? At the end of our cable $x = l$, where the load is located, the load's impedance Z_L is given by

$$Z_L = \frac{V(l,t)}{i(l,t)}. \tag{16}$$

Using (7) and (9), we then have for (16)

$$Z_L = Z_0 \left(\frac{V_1 \, e^{\alpha l + j\beta l} + V_2 \, e^{-\alpha l - j\beta l}}{-V_1 \, e^{\alpha l + j\beta l} + V_2 \, e^{-\alpha l - j\beta l}} \right). \tag{17}$$

Recall that V_1 represents the voltage wave reflected from the load and that V_2 represents the voltage wave incident on the load. Solving for V_2 yields

$$V_2 = e^{2\alpha l + j2\beta l} \, V_1 \frac{Z_L + Z_0}{Z_L - Z_0}. \tag{18}$$

The ratio of the reflected wave to the incident wave at $x = l$ is then given by

[2] See the references for a derivation of L' and C' for a coaxial cable.

$$\frac{V_r(l,t)}{V_i(l,t)} = \frac{V_1 e^{\alpha l} e^{j(\omega t+\beta l)}}{V_2 e^{-\alpha l} e^{j(\omega t-\beta l)}} = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{19}$$

It is easy to show that the ratio of the reflected to the incident current at $x = l$ is given by

$$\frac{i_r(l,t)}{i_i(l,t)} = -\frac{Z_L - Z_0}{Z_L + Z_0}. \tag{20}$$

We will define the useful quantity Γ_L as the load **reflection coefficient**

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{21}$$

Notice that **if** $\Gamma_L = 0$ there is **no** reflected voltage or current. This occurs when the $Z_L = Z_0$ and the load is said to be **impedance matched** to the cable. **This is almost always the situation you desire.** Exceptions are rare. If $\Gamma_L \neq 0$, you will have a portion of our signal reflected back to the source, where of course it can be reflected back to the load, your presumed measuring device. Since the voltage at any point along the cable, including its end, is the superposition of all the voltage waves traveling through it, your measuring device would be then measuring a potentially very complicated waveform rather than just the original waveform sent it. For non-zero Γ_L, the polarity of the reflected voltage will depend on the relative magnitudes of Z_L and Z_0.

Conclusion

Let's summarize. For most situations you will encounter in the lab, you can assume that Z_0 for the cable is real. (This is equivalent to assuming the series resistance of the cable is approximately zero and that the shunt resistance is effectively infinite.) Assume that the voltage pulse has a magnitude of A when incident on the load, whose impedance is assumed to be real, often a good assumption. The table below shows the reflected pulse amplitude for various values of the load impedance.

Termination Resistance R	Reflected Pulse Amplitude
0	$-A$
$0 < R < Z_0$	between 0 and $-A$
Z_0	0
$Z_0 < R < \infty$	between 0 and $+A$
∞	$+A$

Remember, terminate your cables with impedance Z_0 to prevent reflections!

Literature

1. R. Littauer, *Pulse Electronics*, (McGraw-Hill, 1965), pp. 78-85.
2. R.K. Wangsness, *Electromagnetic Fields*, 2nd ed., (John Wiley and Sons, 1986).
3. S. Ramo, J.R. Whinnery, *Fields and Waves in Modern Radio*, 2nd ed., (John Wiley and Sons, 1953).
4. S. Ramo, J.R. Whinnery, T. van Duzer, *Fields and Waves in Communications Electronics*, 2nd ed., (John Wiley and Sons, 1967).

INSTRUCTIONS

1. Set the HP 33120A waveform generator (wfg) to produce a 1.5 MHz square wave with a 1.0 Volt peak-to-peak amplitude. Examine this waveform on the scope. Measure the period T and the frequency f of the square wave. Measure and record the risetime t_r and the falltime t_f of the square wave. Are they the same? Do you expect them to be?

2. The wfg is not quite fast enough to produce pulses short enough for our purposes. We will have to help it using a familiar trick of the trade. The "trick" in this case is simply to differentiate the signal by building a simple RC circuit. See figure 3. You will make the simple RC differentiator by soldering together a capacitor and a resistor. ($R_1 = 250\,\Omega$ and $C_1 = 100\,\text{pF}$.) The instructor will help you with the soldering. Insert the differentiator into the output circuit of the wfg. The resistor R_2 is used to adjust the input impedance of the scope.

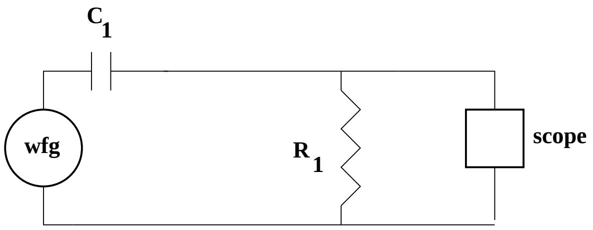

Figure 3: Circuit to generate sharp pulses.

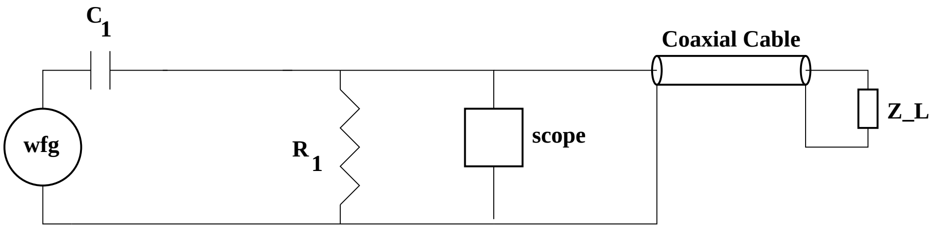

Figure 4: Circuit used to study propagation of waves along a coaxial cable.

3. With the input impedance of the scope $Z_{scope} = 1\ \text{M}\Omega$, observe and sketch the differentiated output of the wfg.

4. Now terminate the scope so that $Z_{scope} = 50\,\Omega$. Observe and sketch the scope trace. Measure t_r and t_f.

5. Calculate C' and L' for the coaxial cable. Use cm as your "unit length." What is your calculated value of Z_0 for the coaxial cable?

6. Attach the the 50 meter long cable to the wfg output/scope input and set $Z_L = \infty$. See figure 4. Observe and sketch the scope trace. You should be able to identify the various sharp spikes that you see. Measure the wave speed v in the cable. How does your measured value for v_{exp} compare with the theoretical value v_{th} given by 10? Also measure the attenuation constant α.

7. Now set $Z_L = 0\,\Omega$. Observe and sketch scope trace. Measure v and α.

8. Determine the cable's Z_0 by minimizing reflections. Use a $1\,\mathrm{k}\Omega$ potentiometer at the cable end.
9. Disconnect the potentiometer at the cable end. Terminate the cable with $50\,\Omega$ using BNC/banana plugs and a $50\,\Omega$ BNC terminator. Observe and sketch scope trace.
10. Add an additional 50 m of cable to the existing one to make a cable about 100 m long or so. Measure v and α for $Z_L = 0\,\Omega$ and $Z_L = \infty$.
11. Determine Z_0 for the long cable using the same technique you used for the 50 m cable. How do the two values compare? Does this make sense?
12. Terminate the long cable with $50\,\Omega$. Observe and sketch the scope trace.
13. Measure v, α, and Z_0 for the 60 m long TV cable.