PHYS 4392 Electromagnetism

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Proof of vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

We seek to prove the vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ found on the inside of the cover of Griffiths and used often in electromagnetism. This is sometimes called the "bac - cab" (pronounced "back minus cab") rule. Using our definition of the cross product from lecture,

$$
\begin{aligned}
[\vec{A} \times (\vec{B} \times \vec{C})]_i &= \epsilon_{ikj} A_j (\vec{B} \times \vec{C})_k & (1) \\
&= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m, & (2)
\end{aligned}
$$

where ϵ_{ikj} is the Levi-Civita tensor. Note that since I already used the suffixes j and k in the first cross product, I need to use something different for the second cross product. I chose l and m. You could have chosen, say, r and s.

A useful identity we will not prove that involves a pair of Levi-Civita tensors is

$$\boxed{\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \tag{3}$$

where,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the Kronecker delta you recall from childhood. Note the order of the indices in the boxed equation. There is a pattern.

Rewriting the RHS of (2) and using (3) yields

$$
\begin{aligned}
A_j \delta_{il} B_l \delta_{jm} C_m - A_j \delta_{jl} B_l \delta_{im} C_m &= A_j B_i C_j - A_j B_j C_i & (4) \\
&= B_i A_j C_j - C_i A_j B_j & (5)
\end{aligned}
$$

Recall that we can shuffle factors like A_j or C_m without thinking since they are just numbers. Shuffling the *indices* of a factor is, in general, a very different story. Examining (5) shows it is nothing more than $\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$. Ergo,

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})} \tag{6}$$