PHYS 5382
Fall 2020
TE Coan
Due: 18 Sep '20 6:00 pm

Homework 3

1. We spent some time on commutation relations. Show that the operator $\hat{C}$ defined through the commutation relation $[\hat{A}, \hat{B}] = i\hat{C}$ is Hermitian if $\hat{A}$ and $\hat{B}$ are.

2. We have seen raising and lowering operators act on kets $|j, m\rangle$ with arbitrary angular momentum j (subject only to the condition that j takes on values that are non-negative multiples of $\hbar/2$). Well, the spin states for our old friends the neutral Ag atom and the electron can also have raising and lowering operators act on them. So, using the familiar basis states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$, verify that for a spin-$\frac{1}{2}$ particle,

(a.)

$$\hat{S}_z = (\hbar/2)\,|+\mathbf{z}\rangle\,\langle+\mathbf{z}| - (\hbar/2)\,|-\mathbf{z}\rangle\,\langle-\mathbf{z}|$$

(b.) and that the raising and lowering operators $\hat{S}_+$ and $\hat{S}_-$, respectively, satisfy

$$\hat{S}_+ = \hbar\,|+\mathbf{z}\rangle\,\langle-\mathbf{z}|$$
$$\hat{S}_- = \hbar\,|-\mathbf{z}\rangle\,\langle+\mathbf{z}|\,.$$

3. Determine the matrix representation of the rotation operator $\hat{R}(\phi\mathbf{k})$ using the states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ as a basis.

(b.) Using your matrix representation, verify that the rotation operator is unitary, that is, it satisfies $\hat{R}^\dagger(\phi\mathbf{k})\hat{R}(\phi\mathbf{k}) = 1$.

4. Determine the column vectors representing the states $|+\mathbf{x}\rangle$ and $|-\mathbf{x}\rangle$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ as a basis.

5. Evaluate $\hat{R}(\theta\mathbf{j})\,|+\mathbf{z}\rangle$, where $\hat{R}(\theta\mathbf{j}) = e^{-i\hat{J}_y\theta/\hbar}$ is the operator that rotates kets counterclockwise by an angle θ about the y axis. Then show that $\hat{R}(\frac{\pi}{2}\mathbf{j})\,|+\mathbf{z}\rangle = |+\mathbf{x}\rangle$. Hint: Express the ket $|+\mathbf{z}\rangle$ as a superposition of the kets $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ and use the fact that $\hat{J}_y\,|\pm\mathbf{y}\rangle = (\pm\hbar/2)\,|\pm\mathbf{y}\rangle$. Then, switch back to the $|\mathbf{z}\rangle$, $|-\mathbf{z}\rangle$ basis.

5. The column vector representing the state $|\Psi\rangle$ is given by

$$|\Psi\rangle \underset{\mathrm{S_z basis}}{\longrightarrow} \frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix}.$$

Using matrix mechanics, show that $|\Psi\rangle$ is properly normalized and calculate the probability that a measurement of S_x yields $\hbar/2$.

(b.) Determine the probability that a measurement of S_y yields $\hbar/2$.

6. Determine the matrix representation of $\hat{J}_x$ in the S_z basis. Hint: Start with the matrix representation of the operator J_x using the states

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle \qquad |-\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle - \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$$

as a basis and then transform to the S_z basis.