PHYS 5383
Spring 2012
TE Coan
Due: 24 Feb '12 6:00 pm

Homework 3

1. The *space reflection* about the origin of the coordinate system is called an "inversion" or a "parity operation." The operation is "discrete" in the sense that you get a specific outcome and not a range of outcomes, like you would with a rotation operator. The parity operator $\hat{P}$ is defined by its operation on kets of position space:

$$\hat{P}\,|\vec{r}\rangle = |-\vec{r}\rangle \qquad \text{and} \qquad \langle\vec{r}|\,\hat{P}^{\dagger} = \langle-\vec{r}|\,.$$

a. Show that the parity operator is Hermitian.

b. What are the eigenvalues of $\hat{P}$? Show your work!

c. A *unitary* operator $\hat{A}$ is one whose inverse $\hat{A}^{-1}$ equals its Hermitian conjugate $\hat{A}^{\dagger}$. Show that $\hat{P}$ is unitary, i.e, $\hat{P}^{\dagger} = \hat{P}^{-1}$.

2. Consider even and odd operators. The operator $\hat{A}$ is said to be of *even* parity if its obeys the condition

$$\hat{P}\hat{A}\hat{P} = \hat{A}$$

and the operator $\hat{B}$ is said to be of *odd* parity if it obeys the condition

$$\hat{P}\hat{B}\hat{P} = -\hat{B}$$

a. Show that even parity operators commute with the parity operator $\hat{P}$ and odd parity operators anticommute with $\hat{P}$.

b. What is the parity of the position operator $\hat{R}$? Recall that $\hat{R}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$.

c. What is the parity of the momentum operator $\hat{p}$? Recall that $\hat{p}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$.

3. Most of the potentials encountered at the microscopic level are symmetric (or "even") with respect to space inversion, $V(-x) = V(x)$ (when written in one dimension). Hence, when $V(x)$ is even, the Hamiltonian $H(x) = -(\hbar^2/2m)d^2/dx^2 + V(x)$ is even also. This implies that the eigenfunctions of $\hat{H}$ are the same as those of the parity operator $\hat{P}$ as long as the eigenvalues of $\hat{H}$ are nondegenerate. Since the eigenstates of $\hat{P}$ have definite parity, the bound states of a particle moving in a one-dimensional symmetric potential have definite parity, either odd or even.

Consider the one-dimensional potential $V(x)$ of the form:

$$V(x) = \begin{cases} -V_0, & \text{for } -a < x < a \\ 0. & \text{otherwise.} \end{cases}$$

We are interested in *bound* states with $E \equiv -|E| < 0$ and of odd parity. (The even ones are done in Griffiths.) Find the transcendental equation that determines the energy eignevalues for odd parity solutions to Schrödinger's equation. Use the variables $k = \sqrt{-2mE}/\hbar$, $q = \sqrt{2m(V_0 + E)}/\hbar$, $y = qa$ and $R = \sqrt{2mV_0 a^2}/\hbar$.

4. This problems is a model of a deuteron, a bound state of a neutron and a proton. In some sense, the deuteron is to nuclear physics what a hydrogen atom is to atomic physics. The potential between the neutron and proton is a function of the radial distance r between them. This is similar to the situation between an electron and proton in the hydrogen atom and we were successful in solving the hydrogen atom, particularly for the case with $l = 0$. We can take the same approach with the deuteron if we make the following modifications to our approach to solving the radial equation. Firstly, we need to use the reduced mass μ of the deuteron:

$$\mu = \frac{m_p m_n}{m_p + m_n} \simeq m_p \simeq 480\,\text{MeV/c}^2$$

Secondly, we need to set $\vec{r} = \vec{r}_1 - \vec{r}_2$ as the relative separation of the two bodies. With these changes, we will model the potential $V(r)$ of the deuteron as

$$V(r) = V(|\vec{r}_1 - \vec{r}_2|) = \begin{cases} -V_0, & \text{if } r \leq a \\ 0. & \text{if } r > a \end{cases}$$

Notice that this potential is similar to the one in the previous problem! We seek solutions to the radial equation for the condition $l = 0$ and that have $u(r = 0) = 0$ since $u(r)$ has no meaning for $r < 0$ and $u(r)$ must be continuous everywhere.

a. What is the solution $u(r)$? Keeping your wits, you should be able to write this guy down without any work if you understood what you did above. Be sure to include normalization constants but don't actually normalize $u(r)$ just yet.

b. Experimentally, it is observed that a gamma ray of energy $E_\gamma = 2.23\,\text{MeV}$ is required to disassociate a deuteron, so it has one bound state with energy $E = -2.23\,\text{MeV}$. We would like to estimate the depth of the well V_0. From scattering information, it can be estimated that the radial extent of the well $a \simeq 2.4$ Fermi. Normalizing $u(r)$ appropriately and remembering the continuity condition for solutions to the Schrödinger equation, show through numerical calculation that $V_0 \simeq 27\,\mathrm{MeV}$. The handy-dandy conversion $\hbar c = 200\,\mathrm{MeV}$-Fermi may be useful.