PHYS 5383
Spring 2012
TE Coan
Due: 5 Apr '12 6:00 pm

Homework 4

1. Recall the Levi-Civita tensor ϵ_{ijk} and its relation to the cross products: $(\mathbf{A}\times\mathbf{B})_i = \epsilon_{ijk}A_jB_k$. See your advanced math book or instructor if this is not familiar to you. You can also see me! Now prove the useful relation

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\sigma\cdot(\mathbf{A}\times\mathbf{B})$$

where σ is any one of the Pauli matrices and **A** and **B** are matrices or operators with directionality (e,g., the momentum operator $\hat{p}$).

2. Consider the spinor $\frac{1}{\sqrt{5}}\binom{2}{1}$. What is the probability that a measurement of $(3S_x+4S_y)/5$ will yield the value $-\hbar/2$? Hint: What are the possible eigenvalues of this operator?

3. Consider the spin state $|1/2\ -1/2\rangle$ built from 2 spins, $s_1 = 3/2$ and $s_2 = 1$. Express $|1/2\ -1/2\rangle$ in terms of $|s_1\, m_1\rangle$ and $|s_2\, m_2\rangle$.

4. Let's construct the operator matrices L_x, L_y, L_+ and L_- for the case of $l = 1$. This is done the same way we did for the case $s = 1/2$. The essential machinery is identical but some details change, like the entries in the various matrices that represent operators. Recall that

$$[L^2, L_z] = 0$$
$$\langle lm'|\, L_z\,|lm\rangle = \hbar\, m\, \delta_{m'm}$$
$$\langle lm'|\, L_\pm\,|lm\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,\delta_{m',m\pm1}$$

Because of the commutation relation, we know that L_z and L^2 will have the same eigenstates. We also know that since $l = 1$, $m \in \{1, 0, -1\}$. This tells us what the eigenvalues for the operator S_z must be for the case $l = 1\,\hbar$. From this you should be able to verify that the matrix operator S_z is

$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

when acting on the eigenspinors

$$|11\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |1\,{-1}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The various matrix elements can be calculated in a straightforward manner. For S_z, the matrix entries are

$$S_z = \begin{pmatrix} \langle 11|\,S_z\,|11\rangle & \langle 11|\,S_z\,|10\rangle & \langle 11|\,S_z\,|1\,{-1}\rangle \\ \langle 10|\,S_z\,|11\rangle & \langle 10|\,S_z\,|10\rangle & \langle 10|\,S_z\,|1\,{-1}\rangle \\ \langle 1\,{-1}|\,S_z\,|11\rangle & \langle 1\,{-1}|\,S_z\,|10\rangle & \langle 1\,{-1}|\,S_z\,|1\,{-1}\rangle \end{pmatrix}$$

For your own understanding, see if you can tell how each row and column entry are labeled by the various m values of the bras and kets. The same scheme was used for the case $S = \hbar/2$ and is used for spins greater than $1\,\hbar$.

a. Determine the matrix form of the operators S_+ and S_- for the case of spin $= 1\,\hbar$. Hint: There will only be 2 non-zero entries for each matrix. Remember to include the value $\hbar$ somewhere appropriately or you will lose points.

b. Find the matrix operators for S_x and S_y, again for the case of spin $= 1\,\hbar$.

5. Paramagnetic Resonance Here is a topic not covered in Griffiths but relates to the behavior of spins in a magnetic field and is closely related to nuclear magnetic resonance. We have already seen that a spin will precess around the direction of a static magnetic field pointing, say, along the z-axis, with a certain angular frequency (the "Larmor" frequency). If we then apply another magnetic field perpendicular to z that rotates with the precessing spin, then this new field would "see" the spin at rest. The component of the spin in the $x-y$ plane will preferentially align opposite to the new field to reach the minimum energy state. Those spins not already aligned with the new field will make a transition to the minimum energy state and in the process emit radiation that can be detected. In the NMR lab in PHYS 4211, your precessing spin is that of a proton, the static B-field is produced by a rare earth magnet and the additional field is produced by a pair of Helmholtz coils that you apply a sinusoidal current to for a short period of time. Let's see how we describe this process quantum mechanically.

Consider an electron in a static B-field of magnitude B_0 pointing in the z-direction and let the electron's only degrees of freedom be its spin states. Suppose further we impose a small oscillating magnetic field $B_1 \cos\omega\, t$ in the x-direction. Schrödinger's equation is then

$$i\hbar\frac{d}{dt}\begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \frac{eg\hbar}{4m_e}\begin{pmatrix} B_0 & B_1\cos\omega\, t \\ B_1\cos\omega\, t & -B_0 \end{pmatrix}\begin{pmatrix}a(t)\\b(t)\end{pmatrix}. \qquad (1)$$

You have already seen this equation when $B_1 = 0$. Look back in Griffiths, example 4.4.2. The symbol g is the g-factor introduced in lecture. For $B_1 = 0$

$$\begin{aligned} i\frac{da(t)}{dt} &= \frac{egB_0}{4m_e}a(t) \equiv \omega_0 a(t) \\ i\frac{db(t)}{dt} &= -\frac{egB_0}{4m_e}b(t) = -\omega_0 b(t) \end{aligned}$$

so that $a(t) = a(0)e^{-i\omega_0 t}$ and $b(t) = b(0)e^{i\omega_0 t}$. You should recognize this solution as $\chi(t) = \binom{a(t)}{b(t}$. Now, for the case of $B_1 \neq 0$, we will introduce $A(t)$ and $B(t)$ as follows:

$$\begin{aligned} A(t) &= a(t)e^{i\omega_0 t} \\ B(t) &= b(t)e^{-i\omega_0 t} \end{aligned} \tag{2}$$

where $a(t)$ and $b(t)$ obey equation (1).

a) Introducing the notation

$$\omega_1 = \frac{egB_1}{4m_e}, \tag{3}$$

derive two first-order differential equations for $A(t)$ and $B(t)$ of the form $i\frac{dA(t)}{dt} = \ldots$ and $i\frac{dB(t)}{dt} = \ldots$. One of the equations will contain $B(t)$ on the RHS and the other will contain $A(t)$.

b) We need to find, somehow, $A(t)$ and $B(t)$ in order to eventually find $a(t0$ and $b(t)$. It is $a(t)$ and $b(t)$ that tell us about the spin orientation of our electron (i.e., what is the probability of measuring spin up and spin down). Rewrite the two equations you found in part a using the "rotating wave" approximation

$$\cos\omega t\, e^{2i\omega_0 t} = \frac{1}{2}(e^{2i\omega_0 t + i\omega t} + e^{2i\omega_0 t - i\omega t}) \simeq \frac{1}{2}e^{i(2\omega_0 - \omega)t}$$

At the so-called "resonance" condition, $\omega = 2\omega_0$, the first term above will oscillate very rapidly and on average contribute nothing. We only need the second term.

One of your equations should include $B(t)$ on the right hand side and the other equation should include $A(t)$.

c) Combine the two first-order equations you derived in part b into a single second-order equation using techniques you learned (or should have learned) in your DIFFEQ math class. You should get an equation of the form

$$\frac{d^2A(t)}{dt^2} = (2\omega_0 - \omega)i\frac{dA(t)}{dt} - \left(\frac{\omega_1}{2}\right)^2 A(t).$$

Solve this guy by trying the solution $A(t) = A(0)e^{i\Omega t}$. Find the two solutions $\Omega_\pm$ for Ω.

d) We can now write

$$A(t) = A_+ e^{i\Omega_+ t} + A_- e^{-i\Omega_- t}.$$

Write a similar equation for $B(t)$. Show your work!

e) With the result from the previous part, we can now finally write solutions for $a(t)$ and $b(t)$:

$$a(t) = A(t)e^{-i\omega_0 t}$$
$$b(t) = B(t)e^{i\omega_0 t}.$$

Suppose at time $t = 0$ the spin is in the "up" state χ_+. This means $a(0) = 1$, $B(0) = 0$, which in turn implies

$$A_+ + A_- = 1$$
$$\Omega_+ A_+ + \Omega_- A_- = 0.$$

These equations can be solved for A_+ and A_-:

$$A_+ = \frac{\Omega_-}{\Omega_- - \Omega_+}$$
$$A_- = -\frac{\Omega_+}{\Omega_- - \Omega_+}$$

Show that the probability $P_-(t)$ for finding the electron in the spin "down" state at time t is given by

$$P_-(t) = \frac{8}{\omega_1^2}\left(\frac{\Omega_+\Omega_-}{\Omega_- - \Omega_+}\right)^2 (1 - \cos(\Omega_- - \Omega_+)t). \tag{4}$$

Deriving this expression is less ferocious than it looks. Think what each entry in the expression for the general spin state $\chi(t) = \binom{a(t)}{b(t)}$ means. Take it from there.

f) At the resonance condition, $\omega = 2\omega_0$, show that equation (4) reduces to

$$P_{res}(t) = \frac{1}{2}(1 - \cos\omega_1 t)$$

So, if you first start with spins in the "up" state, you can flip them into the "down" state by applying the auxiliary B_1 field just long enough. This duration of time is the same as the "π" pulse in the PHYS 4211 NMR lab.