Q1(a).

$$L_+ Y_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\; Y_\ell^{m+1} \qquad (1)$$

$$\Rightarrow Y_4^2 = \frac{1}{\hbar 3\sqrt{2}} L_+ Y_{\ell=4}^{m=1} \qquad (2)$$

Now,

$$L_+ = \hbar e^{i\varphi}\left(\partial/\partial\theta + i\cot\theta\, \partial/\partial\varphi \right) \qquad (3)$$

$$\frac{\partial}{\partial\theta} Y_4^1 = -\frac{3}{8} e^{i\varphi}\sqrt{5/\pi}\,\Big[-\sin\theta\,(-3+7\cos^2\theta)\sin\theta$$

$$+ \cos\theta\,(-14\cos\theta\sin\theta)\sin\theta$$

$$+ \cos\theta\,(-3+7\cos^2\theta)\cos\theta \Big] \qquad (4)$$

$$i\cot\theta\, \partial/\partial\varphi\, Y_4^1 = i\cot\theta\left[-\frac{3}{8}\sqrt{5/\pi}\,\cos\theta\,(-3+7\cos^2\theta)\sin\theta\; i e^{i\varphi}\right]$$

$$= -\cos\theta\left[-\frac{3}{8}\sqrt{5/\pi}\,\cos\theta\,(-3+7\cos^2\theta)\, e^{i\varphi}\right]$$

$$= +\frac{3}{8}\sqrt{5/\pi}\, e^{i\varphi}\cos^2\theta\,(-3+7\cos^2\theta) \qquad (5)$$

So,

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right) Y_4^1 = -\frac{3}{8} e^{i\varphi}\sqrt{5/\pi}\Big[$$

$$-\sin\theta\,(-3+7\cos^2\theta)\sin\theta$$

$$+ \cos\theta\,(-14\cos\theta\sin\theta)\sin\theta \Big]$$

$$= -3/8\, e^{i\varphi}\sqrt{5/\pi}\left[-21\cos^2\theta\sin^2\theta + 3\sin^2\theta \right]$$

$$= -\frac{3}{8} e^{i\varphi}\sqrt{5/\pi}\left[-3\left(7\cos^2\theta - 1\right)\sin^2\theta\right]$$

$$= +\frac{9}{8}\sqrt{5/\pi}\, e^{i\varphi}\left[-1 + 7\cos^2\theta\right]\sin^2\theta \qquad (8)$$

$$\Rightarrow L_+ Y_4^1 = \hbar e^{i\varphi} \frac{9}{8}\sqrt{5/\pi}\, e^{i\varphi}\left[-1 + 7\cos^2\theta\right]\sin^2\theta \qquad (9)$$

SUBSTITUTE (9) INTO (2):

$$\boxed{Y_4^2 = \frac{3}{8} e^{2i\varphi}\sqrt{5/2\pi}\left[-1 + 7\cos^2\theta\right]\sin^2\theta}$$

(b) m SPANS FROM $-l$ to $+l$ IN STEPS OF 1. L_z'S EIGENVALUES ARE EIGENVALUES OF ANGULAR momentum. HENCE

$$\boxed{\text{EIGENVALUES OF } \hat{L}_z = 2\hbar, 1\hbar, 0, -1\hbar, -2\hbar.}$$

(1c) A LITTLE BIT TRICKY / SUBTLE.

NOTE $\frac{3}{5}\hat{L}_x - \frac{4}{5}\hat{L}_y = \underset{\sim}{L}\cdot\hat{n}$, w/ $\hat{n} = \frac{3}{5}\hat{x} - \frac{4}{5}\hat{y}$,

i.e., $\hat{n}$ POINTS IN SOME CRAZY DIRECTION.

LET THIS DIRECTION BE YOUR NEW z DIRECTION. THE EIGENVALUES ARE JUST THE POSSIBLE VALUES OF ANGULAR MOMENTUM YOU CAN MEASURE ALONG THIS DIRECTION. SO, THIS IS NOW JUST LIKE PROBLEM 1 IN pt (b).

$$\boxed{\text{EIGENVALUES} = \{-2\hbar, \hbar, 0, \hbar, 2\hbar\}}$$

Q2 $[L^2, L_z] = 0 \Rightarrow$ COMMON SET OF EIGENSTATES.

NOTE ALSO, $[H, L^2] = [H, L_z] = 0$ SO

$\hat{H}$ & $\hat{L}^2$ & $\hat{L}_z$ HAVE SAME EIGENSTATES.

HENCE

$$H|\alpha\rangle = E|\alpha\rangle$$

$$\left(\frac{L^2}{I} + \alpha L_z\right)|\alpha\rangle = E|\alpha\rangle$$

$$\left(\frac{\ell(\ell+1)\hbar^2}{I} + \alpha m\hbar\right)|\alpha\rangle = E|\alpha\rangle$$

$$\boxed{E = \frac{\ell(\ell+1)\hbar^2}{I} + \alpha m\hbar}$$

Q3. RECALL THAT A WAVEFUNCTION CAN BE EXPRESSED AS A LINEAR SUPERPOSITION OF EIGENSTATES. IN OUR CASE

$$\psi(\text{TRITIUM}) = \sum c_i \varphi_i \text{ (EIGENSTATES OF } He^3)$$

PROBABILITY THAT THE ELECTRON IS IN GROUND STATE OF He^3 IS C_{GS}^2, WHERE C_{GS} IS COEFFICIENT OF φ (GRND STATE OF He^3). NOTE THAT ALL EIGENSTATES OF He^3 ARE ORTHONORMAL. HOW TO COMPUTE C_{GS}? MULTIPLY ABOVE EQN BY φ (GND STATE He^3) AND INTEGRATE OVER ALL SPACE.

$$\int \psi^*(\text{TRITIUM})\, \varphi_{GS}\, d^3r = C_{GS} \int \varphi_{GS}^* \varphi_{GS}\, d^3r$$

$$C_{GS} = \int \psi^*(\text{TRITIUM})\, \varphi_{GS}\, d^3r$$

~~PROB~~ BOTH TRITIUM AND He^3 w/ 1 e^- ARE HYDROGENIC ATOMS. SO, WE KNOW THEIR WAVE FUNCTIONS.

FOR THE GROUND STATE, $n=1$, $l=0$, $m=0$.

$$\psi(\text{TRITIUM}) = R_{10}^{TRI}(r)\, Y_0^0(\theta,\varphi)$$

w/ $R_{10}^{TRI}(r) = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

FOR He^3, w/ $\underline{2}$ PROTONS,

$$\psi(He^3 \text{ GND STATE}) = R_{10}^{He^3}(r)\, Y_0^0(\theta,\varphi)$$

w/ $R_{10}^{He^3}(r) = 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}$

HENCE,

$$C_{GS} = \int R_{10}^{*\,TRI}\, Y_0^0\, R_{10}^{He^3}\, Y_0^0\, d^3r$$

$$= \int_0^\infty R_{10}^{TRI}\, R_{10}^{He^3}\, r^2 dr \int Y_0^0\, Y_0^0\, d\Omega$$

$$= \frac{8\sqrt{2}}{a_0^3} \int_0^\infty e^{-3r/a_0}\, r^2 dr \qquad (5)$$

CHANGE VARIABLES $\quad z \equiv 3r/a_0$

$$\Rightarrow r^2 = \frac{a_0^2}{9} z^2 \quad \& \quad dr = \frac{a_0}{3} dz$$

HENCE

$$C_{GS} = \frac{8\sqrt{2}}{a_0^3} \cdot \frac{a_0^3}{27} \int_0^\infty e^{-z} z^2 dz$$

$$C_{Gs} = \frac{8\sqrt{2}}{27}\left[-e^{-z}\left(z(z+2)+2\right)\right]\Big|_0^{\infty}$$

$$= \frac{16\sqrt{2}}{27} = 0.838$$

$$\boxed{\text{PROB} = C_{Gs}^2 = 70\%}$$

Q4: RECALL FROM LECTURE/TEXTBOOK

(a)
$$E_{n,l=0} = -\left[\frac{m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\Big/ n^2 \qquad (1)$$

$$= E_1/n^2 \qquad n=1,2,3\ldots \qquad (2)$$

$$E_1 = -13.6\ \text{eV}$$

FOR HYDROGEN-LIKE SYSTEMS WITH A NUCLEAR MASS DIFFERENT FROM A PROTON,

$$m \longrightarrow \mu = \frac{m_N m_e}{m_N + m_e} \qquad \text{w/ } m_N = \text{NUCLEAR MASS.}$$

H:

$$\Delta E(2P \to 1S) = E_{2P} - E_{1S}$$

$$= \Delta\left(E_1/n^2\right) = -\frac{13.6\,\text{eV}}{2^2} - \left(-\frac{13.6\ \text{eV}}{1^2}\right)$$

$$= 13.6\,(0.75)\ \text{eV}$$

$$\Rightarrow \boxed{\lambda = 1240/10.2\ \text{nm} = 122\ \text{nm}}$$

(b) FOR DEUTERIUM, $\mu \cong m_e$, SAME FOR H

$$\boxed{\lambda(2P \to 1S) = 122\ \text{nm}}$$

Q4. FOR POSITRONIUM, $\mu = m_e/2$

(c)

$$\Rightarrow E_{n,l=0} = -\frac{1}{2}\left(\frac{m_e}{\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right)\Big/n^2$$

$$= E_1/2n^2$$

$$\Rightarrow \delta E(2P \to 1S) = 5.1\ \text{eV}$$

$$\boxed{\lambda(2P \to 1S) = 2 * 122\ \text{nm} = 244\ \text{nm}}$$

SUMMARIZING, $\lambda(2P \to 1S)$

H	D	e^+e^-
1	1	2