Lecture 11 Review

Van der Pol oscillator & ODE solution via RK4 à la GSL

C++ elements: arrays and `while` statement

Simultaneous 2nd-order ODEs

Suppose we have a pair of <u>simultaneous</u> 2nd-order ODEs.

$$\ddot{x} = f(t, x, y, \dot{x}, \dot{y})$$

$$\ddot{y} = g(t, x, y, \dot{x}, \dot{y})$$

What do we do?

$$d\vec{Y}/dt = \vec{f}(t, \vec{Y})$$ ("standard "form")

$$\vec{Y} = \begin{pmatrix} Y^{(0)}(t) \\ Y^{(1)}(t) \\ \vdots \\ Y^{(N-1)}(t) \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} F^{(0)}(t, \vec{Y}) \\ F^{(1)}(t, \vec{Y}) \\ \vdots \\ F^{(N-1)}(t, \vec{Y}) \end{pmatrix}$$

$$Y^{(0)}(t) = x(t)$$

$$Y^{(1)}(t) = \dot{x}(t)$$

$$Y^{(2)}(t) = y(t)$$

$$Y^{(3)}(t) = \dot{y}(t)$$

Example of Simultaneous 2nd-order ODEs

What to use for $F^{(0)}$, $F^{(1)}$, …?

$$F^{(0)}(t,\vec{Y}) = Y^{(1)}(t) \quad (= \dot{x})$$

$$F^{(2)}(t,\vec{Y}) = Y^{(3)}(t) \quad (= \dot{y})$$

$$F^{(1)}(t,\vec{Y}) = \dot{Y}^{(1)}(t) \quad (= \ddot{x})$$

$$F^{(3)}(t,\vec{Y}) = \dot{Y}^{(3)}(t) \quad (= \ddot{y})$$

If we had more coupled equations, we just add pairs of Y and F as here.

Example: Planetary motion. (See *CP*, sec 15.11)

$$f = -\frac{GMm}{r^2}$$ Attractive force exerted on m by M along center-line.

Take to be the sun

$$\vec{f} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

$$f_x = f\cos\theta = f\,\frac{x}{r} \qquad f_y = f\sin\theta = f\,\frac{y}{r} \qquad r = \sqrt{x^2+y^2}$$

$$\boxed{\frac{d^2x}{dt^2} = -GM\frac{x}{r^3}} \qquad \boxed{\frac{d^2y}{dt^2} = -GM\frac{y}{r^3}}$$

orbit.cc

In class exercise:

Copy and rename rk4.cc → orbit.cc

Modify orbit.cc to accommodate the 2 above simultaneous 2nd-order ODEs.

For simplicity, set GM = 1.

$f[0] = ?$ $f[2] = ?$

$f[1] = ?$ $f[3] = ?$

Initial conditions:

$x(0) = 0.5$ $y(0) = 0.0$ $v_x(0) = 0.0$ $v_y(0) = 1.63$

Adjust number and size of time steps to see orbit close.

Visualize w/ `gnuplot`. (later)

Elements of Orbital Mechanics

Consider the case where a small object (e.g., a comet) orbits the Sun.

$$\vec{F} = -\frac{GMm}{|r|^2}\hat{r}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad \vec{L} = \vec{r} \times (m\vec{v})$$

conserved

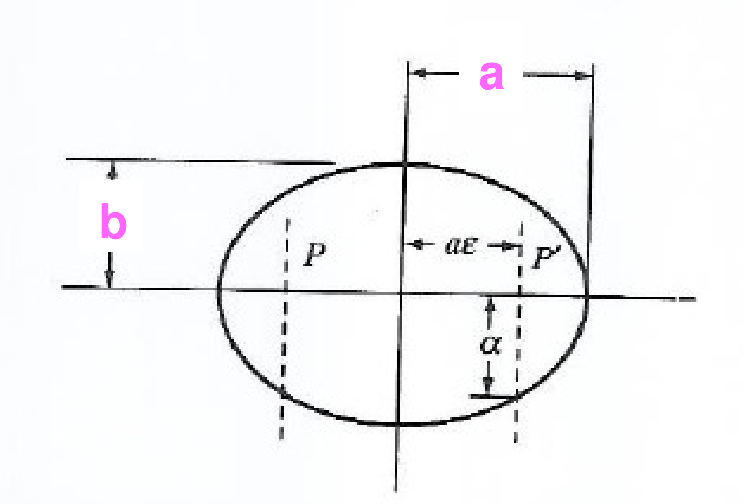
$$\underbrace{\frac{mv^2}{r} = \frac{GMm}{r^2} \qquad v = \sqrt{GM/r} \qquad E = -\frac{GMm}{2r}}_{\text{Circular orbits only}}$$

Circular orbits only

$$e = \sqrt{1 - b^2/a^2}$$

a = “semi-major axis”

b = “semi-minor axis”



$\boxed{\text{Earth\_e} = 0.017}$

Elliptical Orbits and Kepler's 3rd Law

For elliptical orbits around Sun, we have:

$$E = -\frac{GMm}{2a}$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}$$

Exact only for m << M

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3/\mathrm{kg}\text{–}\mathrm{sec}^2$$

$$1\,\mathrm{AU} = 150 \times 10^6\,\mathrm{km} = 1.50 \times 10^{11}\,\mathrm{m}$$

$$1\,\mathrm{yr} = \pi \times 10^7\,\mathrm{sec}$$

$$M_\odot = 2.0 \times 10^{30}\,\mathrm{kg}$$

[r] = AUs

[t] = years

$$GM_\odot = 4\pi^2\,AU^3/yr^2$$

Use this for calc's
w/ orbit.cc

C++ file IO (rd_data_file.cc)

```
/* simple code to read from an input file
   and then write to an output file. */

#include <iostream>
#include <fstream>

using namespace::std;

int main()
{
  ifstream in_file;
  ofstream out_file;

  int r,s,t;
// open inoput and output files:

  in_file.open("myjunk.dat");
  out_file.open("example_out.dat");

// checking file is open for business:
if ( in_file.is_open() )
 {
   while(! in_file.eof() )  // still stuff to read
              {
                in_file >> r >> s >> t;
                out_file << "first col = " << r
                         << "\t 2nd col " << s
                         << "\t 3rd col = " << t << endl;
              }
 }

in_file.close();   // good practice
out_file.close();  // good practice
return 0;
}
```

Summary

Simultaneous 2nd-order ODEs & solution via RK4 à la GSL

Elliptical orbits as an example of simultaneous 2nd-order ODEs

C++ file IO.

Don't suffer in silence. Scream for help!!!