Lecture 12 Review

Simultaneous 2nd-order ODEs & solution via RK4 à la GSL

Elliptical orbits as an example of simultaneous 2nd-order ODEs

C++ file IO.

Simultaneous 2nd-order ODEs

Suppose we have a pair of simultaneous 2nd-order ODEs.

$$\ddot{x} = f(t, x, y, \dot{x}, \dot{y})$$

$$\ddot{y} = g(t, x, y, \dot{x}, \dot{y})$$

What do we do?

$$d\vec{Y}/dt = \vec{f}(t, \vec{Y})$$ ("standard "form")

$$\vec{Y} = \begin{pmatrix} Y^{(0)}(t) \\ Y^{(1)}(t) \\ \vdots \\ Y^{(N-1)}(t) \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} F^{(0)}(t, \vec{Y}) \\ F^{(1)}(t, \vec{Y}) \\ \vdots \\ F^{(N-1)}(t, \vec{Y}) \end{pmatrix}$$

$$Y^{(0)}(t) = x(t)$$

$$Y^{(2)}(t) = y(t)$$

$$Y^{(1)}(t) = \dot{x}(t)$$

$$Y^{(3)}(t) = \dot{y}(t)$$

Example of Simultaneous 2nd-order ODEs

What to use for $F^{(0)}$, $F^{(1)}$, …?

$$F^{(0)}(t,\vec{Y}) = Y^{(1)}(t) \quad (= \dot{x})$$

$$F^{(1)}(t,\vec{Y}) = \dot{Y}^{(1)}(t) \quad (= \ddot{x})$$

$$F^{(2)}(t,\vec{Y}) = Y^{(3)}(t) \quad (= \dot{y})$$

$$F^{(3)}(t,\vec{Y}) = \dot{Y}^{(3)}(t) \quad (= \ddot{y})$$

If we had more coupled equations, we just add <u>pairs</u> of Y and F as here.

Example: Planetary motion. (See *CP*, sec 15.11)

$$f = -\frac{GMm}{r^2}$$

Attractive force exerted on m by M along center-line.

Take to be the sun

$$\vec{f} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

$$f_x = f\cos\theta = f\,\frac{x}{r} \qquad f_y = f\sin\theta = f\,\frac{y}{r} \qquad r = \sqrt{x^2+y^2}$$

$$\frac{d^2x}{dt^2} = -GM\frac{x}{r^3} \qquad \frac{d^2y}{dt^2} = -GM\frac{y}{r^3}$$

Elements of Orbital Mechanics

Consider the case where a small object (e.g., a comet) orbits the Sun.

$$\vec{F} = -\frac{GMm}{|r|^2}\hat{r}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\vec{L} = \vec{r} \times (m\vec{v})$$

conserved

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{GM/r}$$

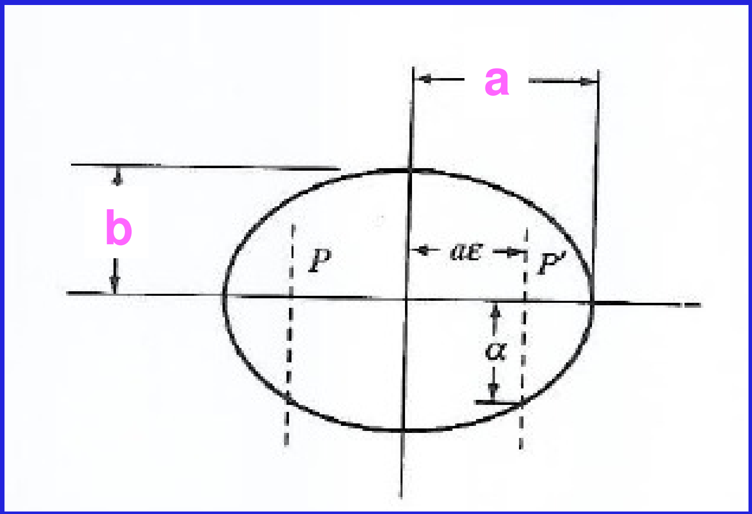
$$E = -\frac{GMm}{2r}$$

Circular orbits only

$$e = \sqrt{1 - b^2/a^2}$$

a = “semi-major axis”

b = “semi-minor axis”



Earth_e = 0.017

Elliptical Orbits and Kepler's 3rd Law

For elliptical orbits around Sun, we have:

$$E = -\frac{GMm}{2a}$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$T^2 = \frac{4\pi^2}{GM}a^3$$

Exact only for m << M

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3/kg{-}sec^2}$$

$$1\,\mathrm{AU} = 150 \times 10^6\,\mathrm{km} = 1.50 \times 10^{11}\,\mathrm{m}$$

$$1\,\mathrm{yr} = \pi \times 10^7\,\mathrm{sec}$$

$$M_\odot = 2.0 \times 10^{30}\,\mathrm{kg}$$

[r] = AUs

[t] = years

$$GM_\odot = 4\pi^2\,AU^3/yr^2$$

Useful for calc's w/ orbit.cc

orbit.cc

In class exercise:

Copy orbit.cc

For simplicity, set GM = 1.

Initial conditions:

$x(0) = 0.5$ $y(0) = 0.0$ $v_x(0) = 0.0$ $v_y(0) = 1.63$

Adjust number and size of time steps to see orbit close.

Visualize w/ `gnuplot`.

Q: Suppose $f = kr^{-n}, \, n \neq 2$ How is orbit effected?

Mercury's orbit shows such an effect. $f = kr^{-2} + k'r^{-4}$

Gen'l Relativity

vdpol.cc

Need a way to use more than 1 parameter in an ODE.

Copy vdpol.cc

```
int
   func (double t, const double y[], double f[],
        void *params)
   {

    struct duo{   // this object holds 2 parameters
           double mu;
           double a;
    };

    duo PARAM;
    PARAM = *(duo *)params;
    //     double mu = *(double *)params;
    f[0] = y[1];
    f[1] = -y[0] - PARAM.mu*y[1]*(y[0]*y[0] - pow(PARAM.a, 2.0)); // CHANGE ME
    return GSL_SUCCESS;
```

C++ `struct` data structure

```
// struct construction

#include <iostream>

using namespace std;

int main()
{
  struct triplet{
            double a;
            double b;
            double c;
};

triplet Test;
            Test.a = 10.0;
            Test.b =15.0;
            Test.c = 30.0;
```

← Defines variable <u>type</u> “triplet” (for the block `double a; double b; double c;`)

← Don’t forget semicolon (;) !! (for `};`)

← Declares variable Test (for `triplet Test;`)

Peculiarites of Non-linear Systems

$$\frac{\Delta N_i}{\Delta t} = \lambda N_i$$ Similar to radioactive decay

limiting population

$$\lambda' = \lambda(N_* - N_i)$$ Growth rate slows as N_* approached.

$$\Rightarrow \frac{\Delta N_i}{\Delta t} = \lambda(N_* - N_i)N_i$$ Typo(s) in book: no λ'

$$N_{i+1} = N_i + \lambda\Delta t(N_* - N_i)N_i$$ "new" = "old" + "change"

$$= N_i(1 + \lambda\Delta t N_*)\left[1 - \frac{\lambda\Delta t}{1+\lambda\Delta t N_*} N_i\right]$$

$$\mu \equiv 1 + \lambda\Delta t N_*$$

$$x_i \equiv \frac{\lambda\Delta t}{\mu} N_i \simeq \frac{N_i}{N_*}$$ $0 \leq x_i \leq 1$

$$x_{i+1} = \mu x_i(1 - x_i)$$ "Logistic map"

lmap.cc

Peculiarites of Non-linear Systems (2)

In class exercise:

Plot x_i versus “generation number i

$$x_{i+1} = \mu x_i (1 - x_i)$$

Try μ = 2.8. 3.3, 3.5. What do you notice?

Summary

More fun with elliptical orbits (orbit.cc)

How to use multiple parameters in an ODE. (vdpol.cc)

C++ `struct` structure

Intro to the logistic map

Don't suffer in silence. Scream for help!!!