Lecture 15 Review

Chaos ID and Lyapunov exponent

First attempt at making a fractal (Sierpiński’s gasket)
Fractals: Sierpinski’s Gasket

For fun, let’s make it first. (sierpin.cc)
Then understand it.
Sierpinski’s Gasket Algorithm

- Draw equilateral triangle and label vertices (1,2,3).
- Randomly pick a single point \( P_0 \) inside triangle.
- Randomly pick an integer from \{1,2,3\}.
- Place 2\(^{nd}\) point halfway btwn \( P_0 \) and vertex from previous step.
- Call this new point \( P_0 \) and repeat last 3 steps.

\[
N \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \left( x_k, y_k \right) + \frac{\left( V x_n, V y_n \right)}{2}
\]

\[n = \text{integer} \left( 1 + 3r_i \right)\]

See sierpin.cc
Fractal: “shape made of parts similar to the whole”

Typical fractal $F$ properties:

- $F$ has structure at arbitrarily small scales.
- $F$ is self-similar. (NB: Not all fractals self-similar.)
- $F$ has non-integral dimension (say what?).

Sierpinski gasket has all 3 properties.
Some Other (Self-Similar) Fractals

Cantor Set:

Koch Curve:
Dimension of Self-Similar Fractals

Q: What do you mean by “dimension”?

- Maybe, number of numbers required to specify location of a point?
- Works OK for line segment and planar shapes. So far, so good.
- Concept fails spectacularly for Koch curve $K$.

$K$ has infinite arc length!

\[
L_0 \sim S_0 \\
L_1 \sim S_1 \\
L_1 = \frac{4}{3} L_0 \\
L_2 = \frac{4}{3} L_1 = \left(\frac{4}{3}\right)^2 L_0 \\
\vdots \\
L_n = \left(\frac{4}{3}\right)^n L_0 \rightarrow \infty \text{ as } n \rightarrow \infty
\]
Q: What do you mean by “dimension”? Examine simple, self-similar structures.

\[ m = r^d \]

\[ d = \frac{\ln m}{\ln r} \]

“similarity” dimension.
Dimension of Koch Curve K

Apply this definition of dimension to K.

$$d = \frac{\ln m}{\ln r}$$

Compare $S_2$ with $S_1$:

- $S_1$ reduced by a factor of 3.
- 4 such identical segments span $S_1$.

$$\Rightarrow \frac{r}{m} = 3$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 4}{\ln 3} = 1.26$$
Apply our definition of dimension.

\[ d = \frac{\ln m}{\ln r} \]

Compare \( S_2 \) with \( S_1 \):
- \( S_1 \) reduced by a factor of 3.
- 2 such identical segments span \( S_1 \).

\[ r = 3 \]
\[ m = 2 \]

\[ d = \frac{\ln m}{\ln r} = \frac{\ln 2}{\ln 3} = 0.63 \]
Q: What is dimension of Sierpinski’s gasket?

$$d = \frac{\ln m}{\ln r}$$

$$r = ?$$

$$m = ?$$

$$d = ?$$
Affine Transformations

Self-similar fractals are generated from *self-affine transformations*:

A mapping one set of points into another using a linear transformation + translation.

\[ x' = Ax + b \]

Translation.

Scaling, shearing or rotation.

S’ski gasket:

\[ (x_{k+1}, y_{k+1}) = \frac{(x_k, y_k) + (Vx_n, Vy_n)}{2} \]

Scaling, Translation
Another (Iterative) Affine Transformation

Barnsley’s fern is a fractal. See fern.cc

\[
(x, y)_{n+1} = \begin{cases} 
(0.5, 0.27y_n) & r < 0.2 \\
(-0.139x_n + 0.263y_n + 0.57, \\
0.246x_n + 0.224y_n - 0.036) & 0.02 \leq r \leq 0.17 \\
(0.17x_n - 0.215y_n + 0.408, \\
0.222x_n + 0.176y_n + 0.0893) & 0.17 < r \leq 0.3 \\
(0.781x_n + 0.034y_n + 0.1075, \\
-0.032x_n + 0.739y_n + 0.27) & 0.3 < r < 1.0 
\end{cases}
\]
Summary

Typical fractal properties.
Meaning of “fractional” dimension, with examples.
Affine transformations and fractal examples.

Don’t suffer in silence. Scream for help!!!