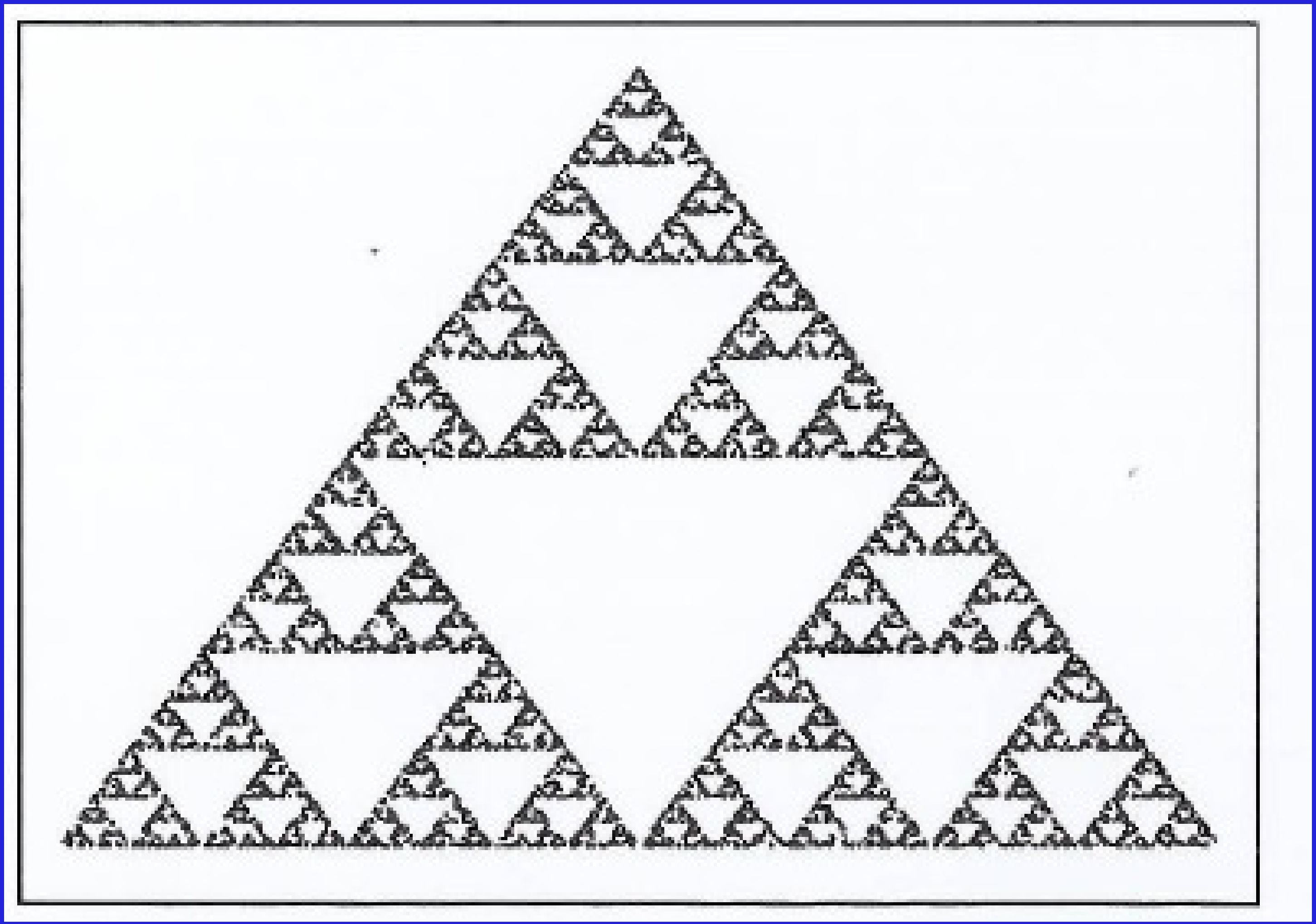
Lecture 15 Review

Chaos ID and Lyapunov exponent

First attempt at making a fractal (Sierpiński's gasket)

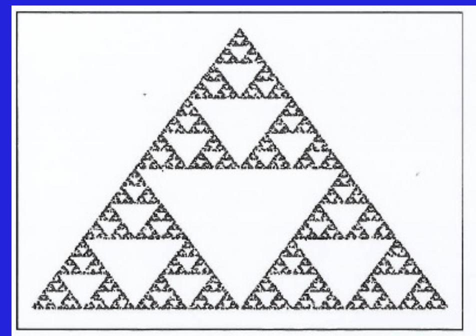
Fractals: Sierpinski's Gasket



For fun, let's make it first. (sierpin.cc)

Then understand it.

Sierpinski's Gasket Algorithm



- Draw equilateral triangle and label vertices (1,2,3).
- Randomly pick a single point P_0 inside triangle.

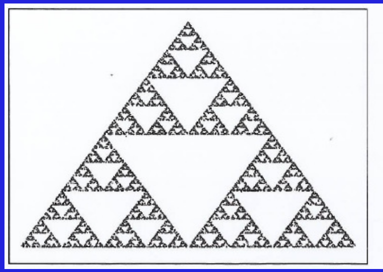
N
- Randomly pick an integer from {1,2,3}.
- Place 2nd point halfway btwn P_0 and vertex from previous step.
- Call this new point P_0 and repeat last 3 steps.

$$(x_{k+1}, y_{k+1}) = \frac{(x_k, y_k) + (Vx_n, Vy_n)}{2}$$

$$n = \text{integer}\,(1 + 3r_i)$$

See sierpin.cc

Typical Fractal Properties



Fractal: "shape made of parts similar to the whole"

Typical fractal F properties:

F has structure at arbitrarily small scales.

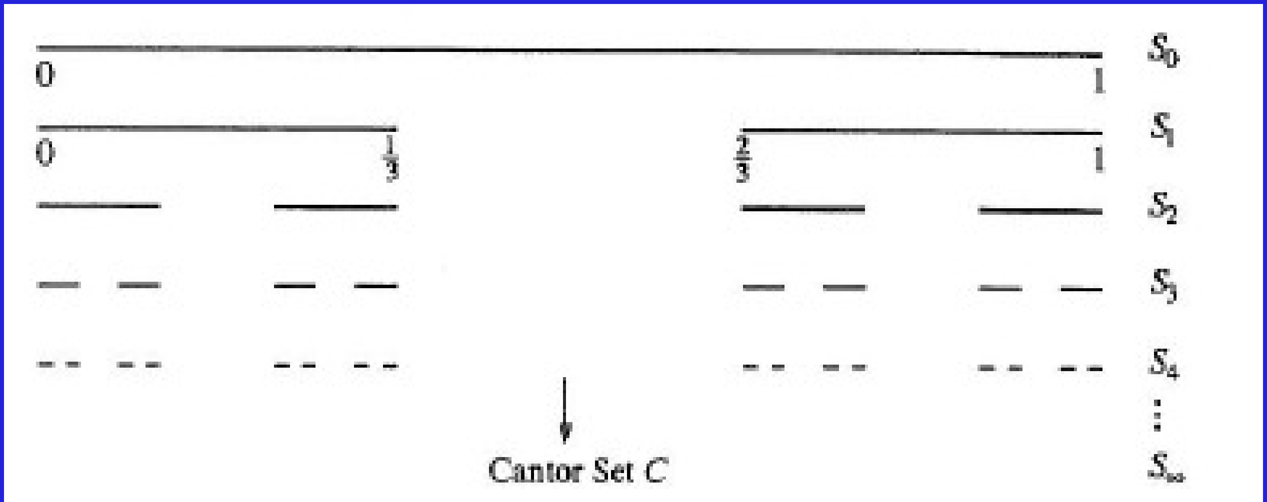
F is self-similar. (NB: Not all fractals self-similar.)

F has non-integral dimension (say what?).

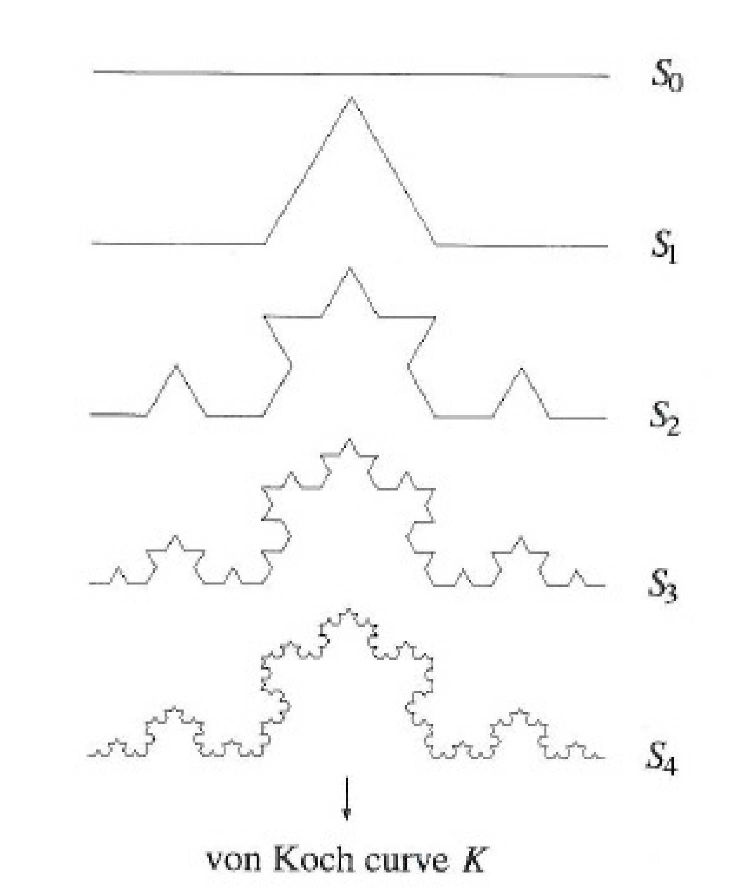
Sierpinski gasket has all 3 properties.

Some Other (Self-Similar) Fractals

Cantor Set:



Koch Curve:



Dimension of Self-Similar Fractals

Q: What do you mean by "dimension"?

- Maybe, number of numbers required to specify location of a point?
- Works OK for line segment and planar shapes. So far, so good.

➢ Concept fails spectacularly for Koch curve K.

K has infinite arc length!

$$L_0 \sim S_0$$

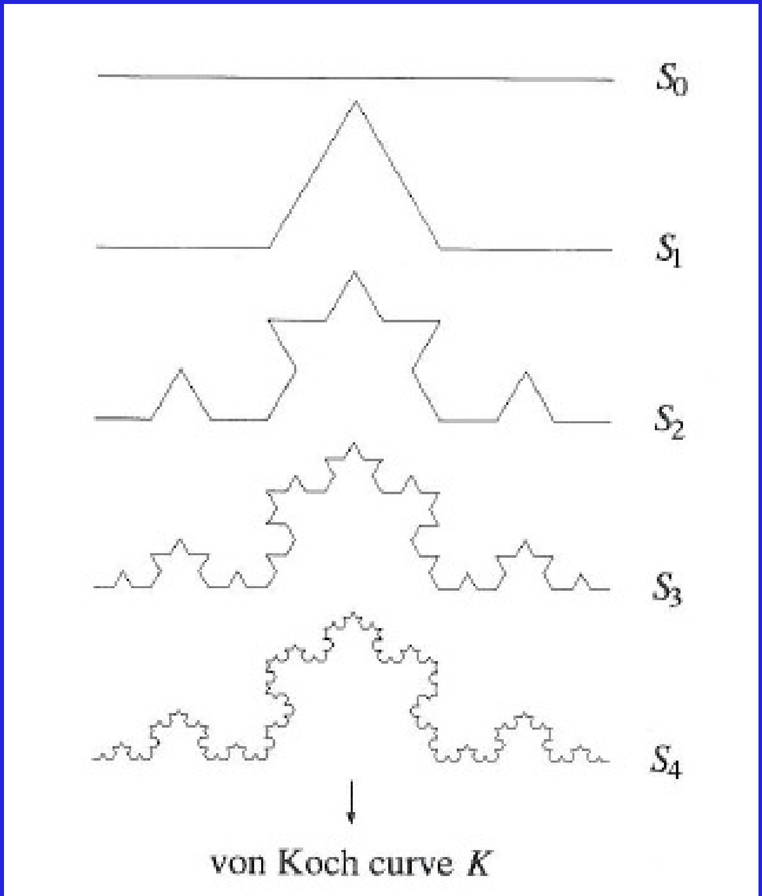
$$L_1 \sim S_1$$

$$L_1 = \frac{4}{3} L_0$$

$$L_2 = \frac{4}{3} L_1 = \left(\frac{4}{3}\right)^2 L_0$$

$$\vdots$$

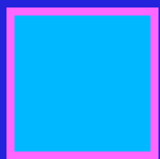
$$L_n = \left(\frac{4}{3}\right)^n L_0 \to \infty \text{ as } n \to \infty$$

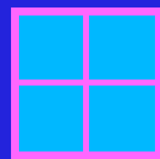


Dimension of Self-Similar Fractals (2)

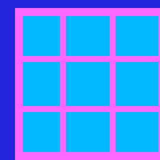
Q: What do you mean by “dimension”?

Examine simple, self-similar structures.

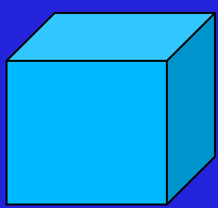


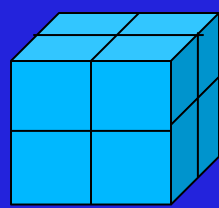


$$m = 4$$
$$r = 2$$



$$m = 9$$
$$r = 3$$





$$m = 8$$
$$r = 2$$

$$m = r^d$$

$$d = \frac{\ln m}{\ln r}$$

“similarity” dimension.

Dimension of Koch Curve K

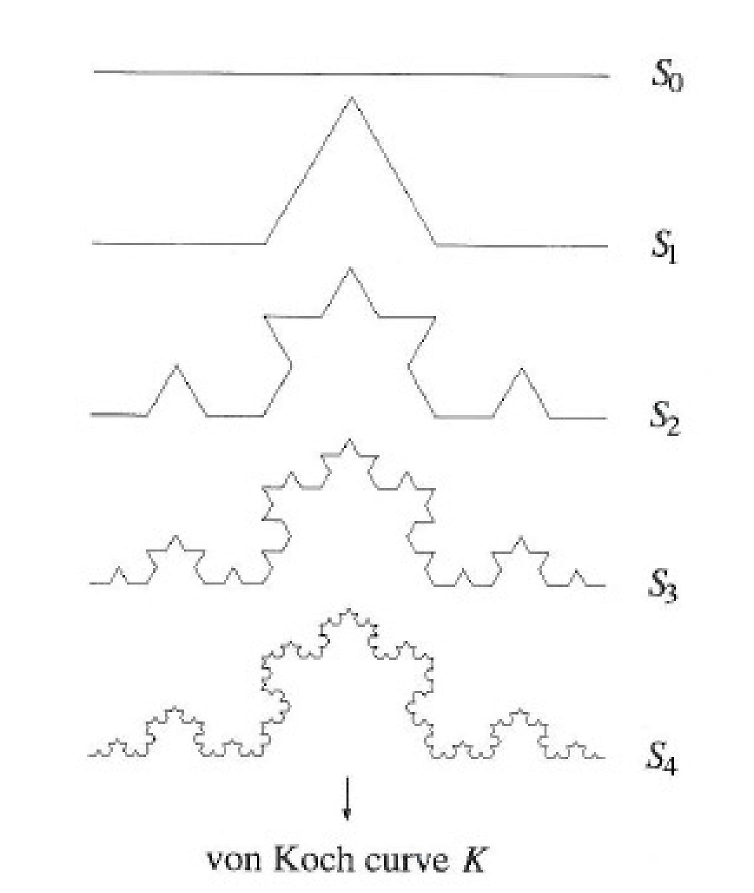
Apply this definition of dimension to K. $d = \frac{\ln m}{\ln r}$

Compare S_2 with S_1:

- S_1 reduced by a factor of 3.
- 4 such identical segments span S_1.

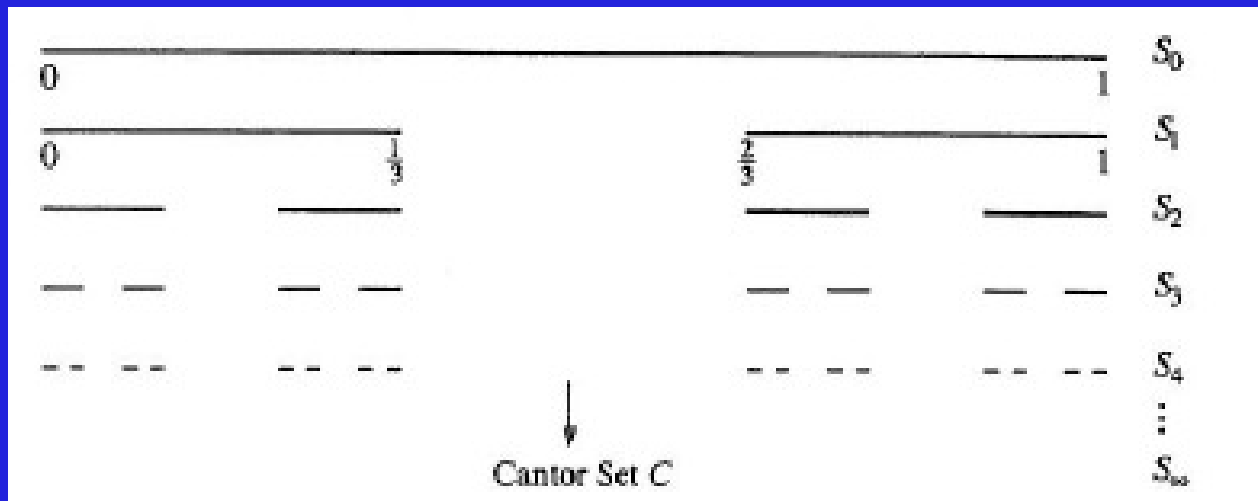
$$\Rightarrow \begin{aligned} r &= 3 \\ m &= 4 \end{aligned}$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 4}{\ln 3} = 1.26$$



Dimension of Cantor Set C

Apply our definition of dimension. $d = \frac{\ln m}{\ln r}$



Compare S_2 with S_1:

- S_1 reduced by a factor of 3.
- 2 such identical segments span S_1.

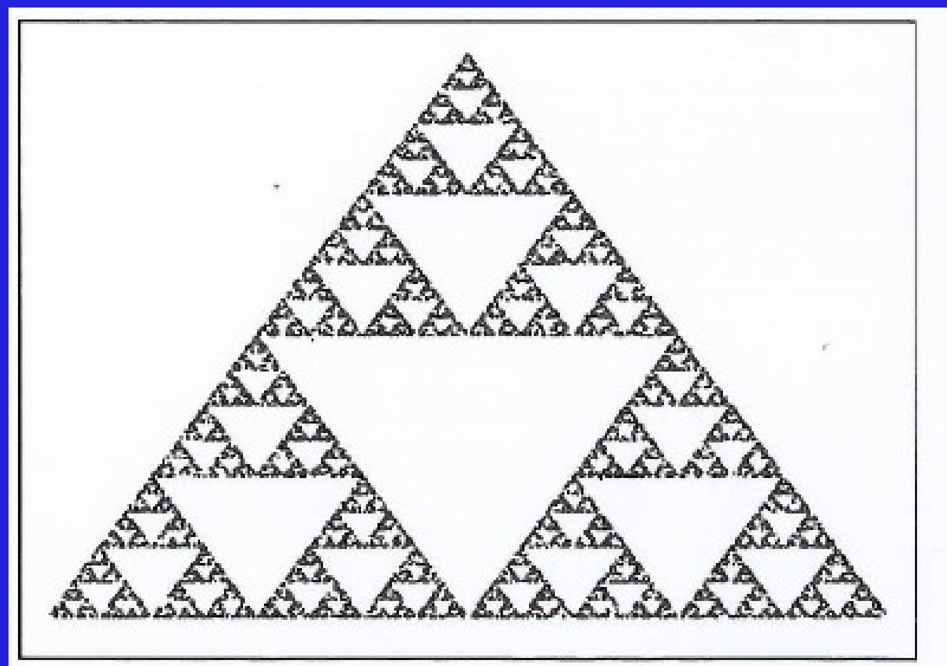
$$\Rightarrow \begin{aligned} r &= 3 \\ m &= 2 \end{aligned}$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 2}{\ln 3} = 0.63$$

Dimension of Sierpinksi Gasket

Q: What is dimension of Sierpinski's gasket?

$$d = \frac{\ln m}{\ln r}$$



r = ?

m = ?

d = ?

Affine Transformations

Self-similar fractals are generated from *self-affine transformations:*

A mapping one set of points into another using a linear transformation + translation.

$$x' = Ax + b$$

Translation.

Scaling, shearing or rotation.

S'ski gasket:

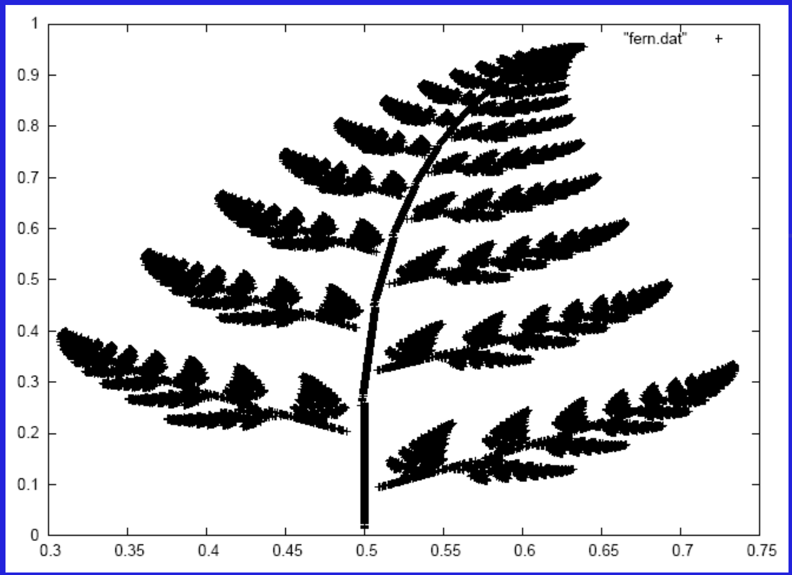
$$(x_{k+1}, y_{k+1}) = \frac{(x_k, y_k) + (Vx_n, Vy_n)}{2}$$

Scaling

Translation

Another (Iterative) Affine Transformation

Barnsley's fern is a fractal. See `fern.cc`



random number

$$
(x, y)_{n+1} = \begin{cases}
(0.5, 0.27y_n) & r < 0.2 \\
(-0.139x_n + 0.263y_n + 0.57, \\ 0.246x_n + 0.224y_n - 0.036) & 0.02 \leq r \leq 0.17 \\
(0.17x_n - 0.215y_n + 0.408, \\ 0.222x_n + 0.176y_n + 0.0893) & 0.17 < r \leq 0.3 \\
(0.781x_n + 0.034y_n + 0.1075, \\ -0.032x_n + 0.739y_n + 0.27) & 0.3 < r < 1.0
\end{cases}
$$

Summary

Typical fractal properties.

Meaning of “fractional” dimension, with examples.

Affine transformations and fractal examples.

Don’t suffer in silence. Scream for help!!!