Lecture 21 Review

DFT (theory and lab example).
Fourier Transforms (Reprise)

Fourier series great for periodic functions (work for non-periodic, tooy function).
Fourier “transforms” great for non-periodic functions.

\[
H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i ft} \, dt
\]

\[
h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i ft} \, df
\]

our choice

H(f) is a measure of the contribution of a particular frequency to f (t) .
Fourier Transforms (2)

We can think of “time space” and “frequency space.”
h(t) and H(f) have various symmetry properties:

- h(t) even, H(f) even
- h(t) odd, H(f) odd
- h(t) real, H(-f) = [H(f)]*
- h(t) imaginary, H(-f) = - [H(f)]*

Parseval’s Theorem:

"total power" $\equiv \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$

"one-sided power spectral density" (PSD)

$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \quad 0 \leq f < \infty$
Discrete Fourier Transform (Reprise)

**Issue:** Often discretely sample a waveform and need to know: its shape and/or its frequency characteristics.

Sample (i.e., measure w/ an instrument) every \( \Delta \) seconds. (1/\( \Delta \) is the “sampling rate”)

\[
\cos(2\pi f t) \quad f = 2
\]
sampled every 1 second

“Nyquist critical frequency” \( f_c \equiv \frac{1}{2\Delta} \)
Sampling Theorem and Aliasing

“Nyquist critical frequency” \[ f_c \equiv \frac{1}{2\Delta} \]

“Good news:”

- Continuous h(t) samples at intervals \( \Delta \)
- “bandwidth” limited to frequencies < \( f_c \), i.e., \( H(f) = 0 \ \forall \ |f| > f_c \)

\[ h(t) \text{ completely determined by } h_n . \]

\[ h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t-n\Delta)]}{\pi(t-n\Delta)} \]

“Bad news:”

- If \( h(t) \) not bandwidth limited to \( f < f_c \)
- All of PSD outside \(-f_c < f < f_c\)
- folded into \(-f_c < f < f_c\) ("aliasing")
Sampling Theorem and Aliasing (2)

Limit sampled frequencies to $< f_c$
Discrete Fourier Transform (Review)

Let

\[ h_k \equiv h(t_k) \quad t_k \equiv k\Delta \quad k = 0, 1, 2 \ldots, N - 1 \]

\[ f_n \equiv \frac{n}{N\Delta} \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} \]

\( N = \) number of data points
\( \Delta = \) time between data points.

Discretize integral form of FT:

\[ H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} \, dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N} \]

\[ H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N} \]

DFT

\[ H(f_n) \approx \Delta H_n \]

continuous v. discrete FTs

\[ h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i kn/N} \]

Inverse DFT

\[ \sum_{k=0}^{N-1} |h_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2 \]

Discrete form of Parseval’s theorem
Discrete Fourier Transform

\[ H(f) = \int_{-\infty}^{\infty} h(t)e^{2\pi if t} \, dt \]

\[ h(t) = \int_{-\infty}^{\infty} H(f)e^{-2\pi if t} \, df \]

make_fourier_data.cc
dft.cc
inv_dft.cc

\text{Sin}(2\pi f t): \text{where is peak } n \text{ in } H_n?
Fast Fourier Transform

DFT is **sloooow**. Execution time $\propto N^2$.

Compare $N = 1000$ to $N = 5000$

Fast Fourier Transform (FFT) to the rescue.

**Not** a new type of transform, but a new way to calculate FT.

$$W \equiv e^{2\pi i/N} \quad \text{“twiddle factor”}$$

$$H_n \equiv \sum_{k=0}^{N-1} W^n h_k$$

$$F_k = \sum_{j=0}^{N-1} e^{2\pi i j k/N} f_j$$

$$= \sum_{j=0}^{N/2-1} e^{2\pi i (2j) k/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i (2j+1) k/N} f_{2j+1}$$

$$= \sum_{j=0}^{N/2-1} e^{2\pi i j k/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi i j k/(N/2)} f_{2j+1}$$

repeat...$$= F^e_k + W^k F^o_k \quad 0 \leq k \leq N - 1$$

Execution time $\propto N \log_2 N$
Fast Fourier Transform (2)

make_fourier_data.cc
gsl_fft.cc
gsl_inv_fft.cc

GSL FFT routines store $\text{Imag}(H_n)$ differently from DFT $H(k) = H(N-k)^*$ (Not all $2N$ $H(k)$ numbers independent.)

- Compare DFT output w/ FFT output.
- Compare execution times: DFT v. FFT w/ $N = 5000$
### FFT (3)

**Square Wave Data**

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<th>dft.dat</th>
<th>gsl_fft.dat</th>
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**Why the relative minus (-) sign for Imag H(n)?**

*Inefficient storage*
Summary

DFT Review.
FFT (theory and exercises).

Don’t suffer in silence. Scream for help!!!