Lecture 21 Review

DFT (theory and lab example).

Fourier Transforms (Reprise)

Fourier series great for periodic functions (work for non-periodic, tooy function).

Fourier "transforms" great for non- periodic functions.

FT → $H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$

Inverse FT → $h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$

our choice

H(f) is a measure of the contribution of a particular frequency to f (t) .

Fourier Transforms (2)

We can think of ‘time space” and “frequency space.”

h(t) and H(f) have various symmetry properties:

h(t) even	H(f) even
h(t) odd	H(f) odd
h(t) real	H(-f) = [H(f)]*
h(t) imaginary	H(-f) = - [H(f)]*

Parseval’s Theorem:

$$\text{"total power"} \equiv \int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

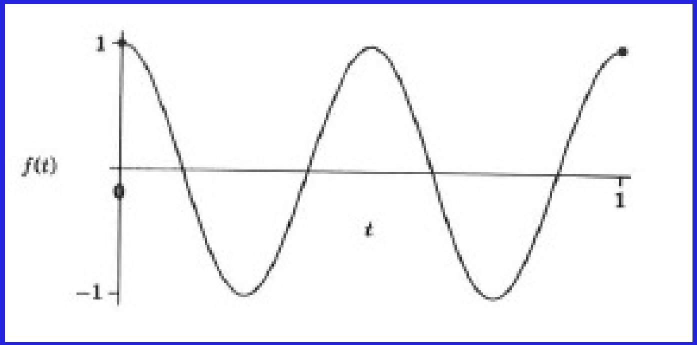
“one-sided power spectral density” (PSD)

$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \quad 0 \leq f < \infty$$

Discrete Fourier Transform (Reprise)

Issue: Often discretely sample a waveform and need to know: its shape and/or its frequency characteristics.

Sample (i.e., measure w/ an instrument) every Δ seconds.
(1/Δ is the "sampling rate")



cos(2πf t)   f = 2
sampled every 1 second

"Nyquist critical frequency" $f_c \equiv \frac{1}{2\Delta}$

Sampling Theorem and Aliasing

"Nyquist critical frequency" $f_c \equiv \frac{1}{2\Delta}$

"Good news:"

- Continuous h(t) samples at intervals Δ
- "bandwidth" limited to frequencies < f_c , i.e., $H(f) = 0 \; \forall \; |f| > f_c$

$\boxed{\text{h(t) completely determined by } h_n \text{ .}}$

$$h(t) = \Delta \sum_{n=-\infty}^{n=+\infty} h_n \frac{\sin[2\pi f_c (t-n\Delta)]}{\pi(t-n\Delta)}$$

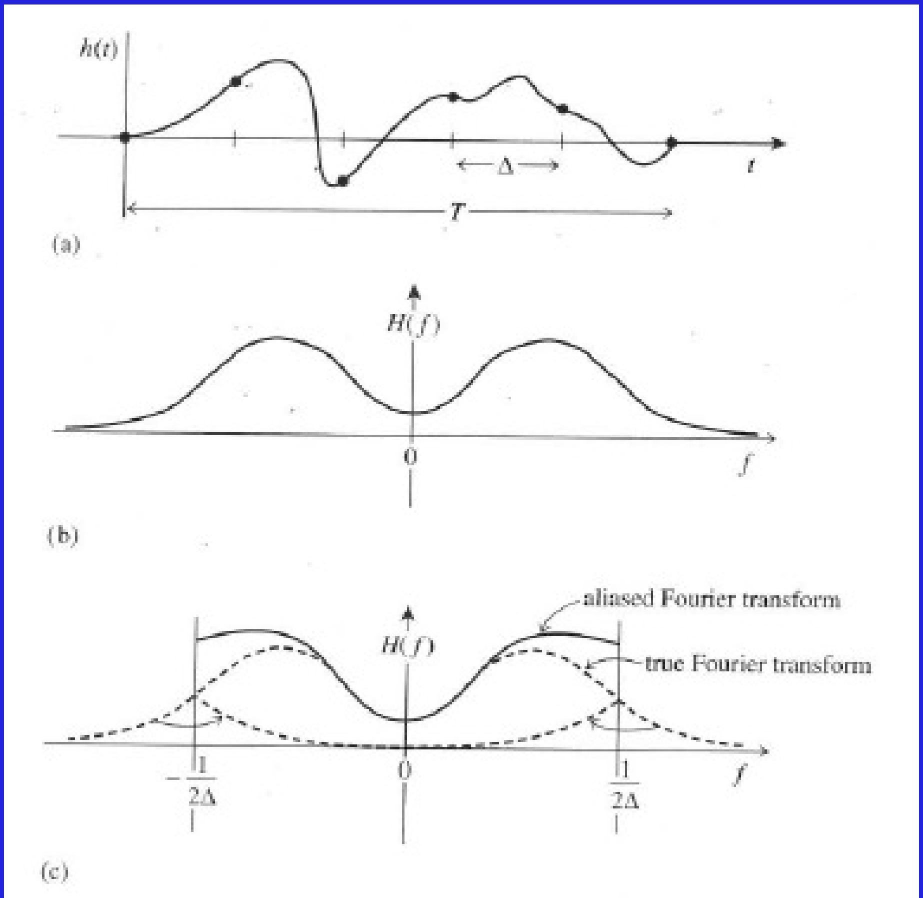
"Bad news:"

If h(t) not bandwidth limited to $f < f_c$

All of PSD outside $-f_c < f < f_c$

folded into $-f_c < f < f_c$ ("aliasing")

Sampling Theorem and Aliasing (2)



Limit sampled frequencies to < f_c

Discrete Fourier Transform (Review)

Let

$$h_k \equiv h(t_k) \qquad t_k \equiv k\Delta \qquad k = 0, 1, 2 \ldots, N-1$$

$$f_n \equiv \frac{n}{N\Delta} \qquad n = -\frac{N}{2}, \; \ldots \; \frac{N}{2}$$

N = number of data points

Δ = time between data points.

Discretize integral form of FT:

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t}\, dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \underbrace{\sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}}_{\text{DFT}}$$

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}$$ DFT

$$H(f_n) \approx \Delta H_n$$ continuous v. discrete FTs

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}$$ Inverse DFT

$$\sum_{k=0}^{N-1} |h_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2$$ Discrete form of Parseval's theorem

Discrete Fourier Transform

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$$

```
make_fourier_data.cc

dft.cc

inv_dft.cc
```

Sin(2πf t): where is peak n in H_n?

Fast Fourier Transform

DFT is slooow. Execution time ∝ N*N.

Compare N = 1000 to N = 5000

Fast Fourier Transform (FFT) to the rescue.

Not a new type of transform, but a new way to calculate FT.

$$W \equiv e^{2\pi i/N}$$ "twiddle factor"

$$H_n \equiv \sum_{k=0}^{N-1} W^{nk} h_k$$

$$F_k = \sum_{j=0}^{N-1} e^{2\pi i jk/N} f_j$$

$$= \sum_{j=0}^{N/2-1} e^{2\pi i(2j)k/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i(2j+1)k/N} f_{2j+1}$$

$$= \sum_{j=0}^{N/2-1} e^{2\pi i jk/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi i jk/(N/2)} f_{2j+1}$$

repeat… $$= F_k^e + W^k F_k^o$$ $0 \le k \le N - 1$

Execution time ∝ N $\log_2$ N

Fast Fourier Transform (2)

```
make_fourier_data.cc
gsl_fft.cc
gsl_inv_fft.cc
```

GSL FFT routines store Imag(Hn) differently from DFT
H(k) = H(N-k)* (Not all 2N H(k) numbers independent.)

- Compare DFT output w/ FFT output.
- Compare execution times: DFT v. FFT w/ N = 5000

FFT (3)

square wave data

dft.dat

0 33 0
1 -27.3487
1.72063
2 13.846 -1.74915
3 -0.32786
0.0625427
4 -6.52502
1.67534

inefficient storage

5 5.41695 -
1.76007
6 -0.311563
0.123357

Why the relative minus (-) sign for Imag H(n) ?

gsl_fft.dat

0	33
1	-27.3487
2	-1.72063
3	13.846
4	1.74915
5	-0.32786
6	-0.0625427
7	-6.52502
8	-1.67534
9	5.41695
10	1.76007
11	-0.311563
12	-0.123357

Summary

DFT Review.

FFT (theory and exercises).

Don’t suffer in silence. Scream for help!!!