Lecture 24 Review

- Gaussian elimination.
- LU decomposition

Linear Algebra Highlights

- Physics often requires solution of eiegnevectors & eigenvalues.
  e.g., normal modes, eigenfrequencies, bound energy states, ……

Equations of the form: $\boxed{Ax = \lambda\, x}$ Solve for λ & x, A is given.
Nonlinear equation.

$\boxed{(A - \lambda I)x = 0}$ Eigenvectors x lie in the "nullspace" of A - λI

For λ to be an eigenvalue of A:

1) non-zero x for which Ax = λx
2) A - λI is singular
3) det(A - λI) = 0

Each is necessary and sufficient

#3 implies sum of n eigenvalues of A = sum of diagonal entries of A

$$\begin{Vmatrix} (a_{11}-\lambda) & a_{12} & a_{13} \\ a_{21} & (a_{22}-\lambda) & a_{23} \\ a_{31} & a_{32} & (a_{33}-\lambda) \end{Vmatrix} = (a_{11}-\lambda)[(a_{22}-\lambda)(a_{33}-\lambda)-a_{32}a_{23}] + \cdots = 0$$

Eigenvalues

#3 also implies:

If A is triangular (lower <u>or</u> upper), λ's appear on diagonal of A

$$\begin{Vmatrix} (a_{11}-\lambda) & a_{12} & a_{13} \\ 0 & (a_{22}-\lambda) & a_{23} \\ 0 & 0 & (a_{33}-\lambda) \end{Vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

Now, suppose the n X n matrix A has n linearly independent eigenvectors

Then if you write them as the column vectors of a matrix S

$S^{-1}AS$ is diagonal w/ the λ's of A along the diagonal:

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Eigenvalues (2)

Here's why $S^{-1}AS = \Lambda$

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$AS = S\Lambda$ or $S^{-1}AS = \Lambda$

"similarity transformation"

Furthermore, for any non-singular matrix M:
If $B = M^{-1}AM$ then A & B have same λ's w/ same multiplicities.

Here's why:

$$\det(B - \lambda I) = \det(M^{-1}AM - \lambda I) = \det(M^{-1}(A - \lambda I)M)$$
$$= \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I)$$

Strategy for Finding λ and x

Strategy to find λ's and eigenvectors x of A.

Perform similarity transformations to diagonalize A

$P^{-1}AP = A'$ ← Diagonal (A and A' have same eigenvalues)

$P \equiv P_1 P_2 P_3 \ldots$ P could be a product of many transformations.

$$P^{-1}AP = \ldots P_3^{-1} P_2^{-1} (P_1^{-1} A P_1) P_2 P_3 \ldots$$

Amazingly, we can get eigenvectors from P

Suppose the n eigenvectors u_i of A are linearly indpt and are a basis.

$$v_i \equiv P^{-1} u_i$$

$$A' v_i = (P^{-1}AP)(P^{-1}u_i) = P^{-1}Au_i = \lambda v_i$$

$$v_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Only non-zero term

Now note: $Pv_i = P(P^{-1}u_i) = u_i$

eigenvalues of A = columns of P !

Examples of P-type Transformations

$$P = \begin{bmatrix} 1 & & & & \\ \ldots & \cos\theta & \ldots & \sin\theta & \\ & \vdots & 1 & \vdots & \\ & -\sin\theta & \ldots & \cos\theta & \\ & & & & 1 \end{bmatrix}$$

Works on some elements of A

Seek to eliminate off-diagonal terms

Also sometimes useful to factorize A: A = PQ

then note: QP = $(P^{-1} A)P$

similarity transformation

Octave Calisthenics

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}$$

Q: What is the sum S of the eigenvalues?

Q: What are the eigenvalues? Solve first by hand, then by `octave`.

FYI: octave:19> `help -i eig` will be helpful !!

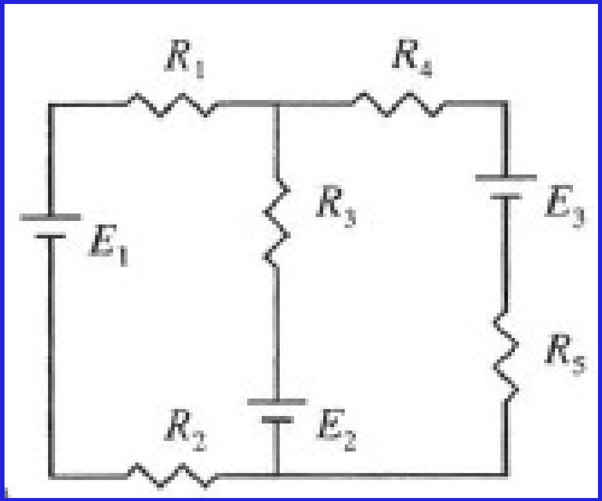
Q: What is the LU decomposition of A?

Q: What do you notice about UL ? Why is this?

Q: What are the eigenvectors of A ?

Q: What is A^{-1} ? Verify this.

Octave Calisthenics (2)



R1 = R2 = 1Ω	E1 = 2V
R3 = R4 = 2Ω	E3 = 5V
R5 = 5Ω	E2 = 10V

Find I in all legs.

Summary

Finding eigenvalues and eigenvectors

`octave` calisthenics

Don’t suffer in silence. Scream for help!!!