Lecture 24 Review

- Gaussian elimination.
- LU decomposition
Linear Algebra Highlights

- Physics often requires solution of eigenvectors & eigenvalues. e.g., normal modes, eigenfrequencies, bound energy states, ……

Equations of the form: \[ Ax = \lambda x \] Solve for \( \lambda \) & \( x \), \( A \) is given. Nonlinear equation.

\[(A - \lambda I)x = 0\] Eigenvectors \( x \) lie in the “nullspace” of \( A - \lambda I \)

For \( \lambda \) to be an eigenvalue of \( A \):
1) non-zero \( x \) for which \( Ax = \lambda x \)
2) \( A - \lambda I \) is singular
3) \( \det(A - \lambda I) = 0 \)

Each is necessary and sufficient

#3 implies sum of n eigenvalues of \( A \) = sum of diagonal entries of \( A \)

\[
\begin{vmatrix}
(a_{11} - \lambda) & a_{12} & a_{13} \\
a_{21} & (a_{22} - \lambda) & a_{23} \\
a_{31} & a_{32} & (a_{33} - \lambda)
\end{vmatrix} = (a_{11} - \lambda)[(a_{22} - \lambda)(a_{33} - \lambda) - a_{32}a_{23}] + \cdots = 0
\]
If $A$ is triangular (lower or upper), $\lambda$'s appear on diagonal of $A$

\[
\begin{vmatrix}
  (a_{11} - \lambda) & a_{12} & a_{13} \\
  0 & (a_{22} - \lambda) & a_{23} \\
  0 & 0 & (a_{33} - \lambda)
\end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0
\]

Now, suppose the $n \times n$ matrix $A$ has $n$ linearly independent eigenvectors

Then if you write them as the column vectors of a matrix $S$

$S^{-1}AS$ is diagonal w/ the $\lambda$'s of $A$ along the diagonal:

\[
S^{-1}AS = \Lambda = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
\]
Eigenvalues (2)

Here's why $S^{-1}AS = \Lambda$

$$AS = A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$AS = S\Lambda$ or $S^{-1}AS = \Lambda$

“similarity transformation”

Furthermore, for any non-singular matrix $M$:
If $B = M^{-1}AM$ then $A$ & $B$ have same $\lambda$’s w/ same multiplicities.

Here's why:

$$\det(B - \lambda I) = \det(M^{-1}AM - \lambda I) = \det(M^{-1}(A - \lambda I)M)$$

$$= \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I)$$
Strategy for Finding $\lambda$ and $x$

Strategy to find $\lambda$'s and eigenvectors $x$ of $A$.

Perform similarity transformations to diagonalize $A$

$$P^{-1}AP = A'$$

$P \equiv P_1P_2P_3 \ldots$ P could be a product of many transformations.

$$P^{-1}AP = \ldots P_3^{-1}P_2^{-1}(P_1^{-1}AP_1)P_2P_3\ldots$$

Amazingly, we can get eigenvectors from $P$

Suppose the $n$ eigenvectors $u_i$ of $A$ are linearly indpt and are a basis.

$$v_i \equiv P^{-1}u_i$$

$$A'v_i = (P^{-1}AP)(P^{-1}u_i) = P^{-1}Au_i = \lambda v_i$$

$$v_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Only non-zero term

Now note: $Pv_i = P(P^{-1}u_i) = u_i$

Eigenvectors of $A$ are columns of $P$!
Examples of P-type Transformations

$$P = \begin{bmatrix} 1 \\ \vdots \\ \cos \theta & \ldots & \sin \theta \\ \vdots \\ -\sin \theta & \ldots & \cos \theta \\ 1 \end{bmatrix}$$

Works on some elements of $A$
Seek to eliminate off-diagonal terms

Also sometimes useful to factorize $A$: $A = PQ$
then note: $QP = (P^{-1} A)P$

similarity transformation
Octave Calisthenics

\[ A = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix} \]

Q: What is the sum \( S \) of the eigenvalues?
Q: What are the eigenvalues? Solve first by hand, then by \texttt{octave}.
FYI: \texttt{octave:19> help -i eig} will be helpful!!
Q: What is the LU decomposition of \( A \)?
Q: What do you notice about \( UL \)? Why is this?
Q: What are the eigenvectors of \( A \)?
Q: What is \( A^{-1} \)? Verify this.
Octave Calisthenics (2)

\[ R_1 = R_2 = 1\,\Omega \quad E_1 = 2\,V \]
\[ R_3 = R_4 = 2\,\Omega \quad E_3 = 5\,V \]
\[ R_5 = 5\,\Omega \quad E_2 = 10\,V \]

Find I in all legs.
Summary

Finding eigenvalues and eigenvectors

calisthenics