Lecture 12 Review

Simultaneous 2nd-order ODEs & solution via RK4 à la GSL

Elliptical orbits as an example of simultaneous 2nd-order ODEs

C++ file IO.

Return of the van der Pol oscillator

Our favorite nonlinear oscillator.

$$\ddot{x} + \mu(x^2 - a^2)\dot{x} + \omega_0^2 x = 0$$

(μ is small and positive)

$\mu = 0$: familiar simple harmonic motion (SHM)

$x^2 - a^2 > 0$: damped SHM

$x^2 - a^2 < 0$: “anti-damped” SHM

Note $x^2 - a^2$ reverses sign as x evolves with time.

Equilibrium motion of x results. (You saw this for yourself last time.)

vdpol.cc

Need a way to use <u>more</u> than 1 parameter in an ODE.

Copy vdpol.cc

```
int
    func (double t, const double y[], double f[],
          void *params)
    {

      struct duo{    // this object holds 2 parameters
        double mu;
        double a;
      };

      duo PARAM;
      PARAM = *(duo *)params;
      //       double mu = *(double *)params;
      f[0] = y[1];
      f[1] = -y[0] - PARAM.mu*y[1]*(y[0]*y[0] - pow(PARAM.a, 2.0)); // CHANGE ME
      return GSL_SUCCESS;
```

C++ `struct` data structure

```
// struct construction

#include <iostream>

using namespace std;

int main()
{
  struct triplet{
       double a;
       double b;
       double c;
};

triplet Test;
       Test.a = 10.0;
       Test.b =15.0;
       Test.c = 30.0;
```

Defines variable <u>type</u> “triplet”

Don’t forget semicolon (;) !!

Declares variable Test

Peculiarities of Non-linear Systems

$$\frac{\Delta N_i}{\Delta t} = \lambda' N_i$$ Similar to radioactive decay

limiting population

$$\lambda' = \lambda(N_* - N_i)$$ Growth rate slows as N_* approached.

$$\Rightarrow \frac{\Delta N_i}{\Delta t} = \lambda(N_* - N_i)N_i$$ Typo(s) in book: no λ'

$$N_{i+1} = N_i + \lambda\Delta t(N_* - N_i)N_i$$ "new" = "old" + "change"

$$= N_i(1 + \lambda\Delta t N_*)\left[1 - \frac{\lambda\Delta t}{1+\lambda\Delta t N_*}N_i\right]$$

$$\mu \equiv 1 + \lambda\Delta t N_*$$

μ is a kind of growth parameter

$\lambda\Delta t$ is the number of insects born/generation

$$x_i \equiv \frac{\lambda\Delta t}{\mu}N_i \simeq \frac{N_i}{N_*}$$ $0 \le x_i \le 1$

$$\boxed{x_{i+1} = \mu x_i(1 - x_i)}$$

"Logistic map"

lmap.cc

Peculiarities of Non-linear Systems (2)

In class exercise:

Plot x_i versus “generation number i

$$x_{i+1} = \mu x_i (1 - x_i)$$

Try μ = 0.75, 2.8. 3.3, 3.5. What do you notice?

Fixed points and 1D maps

$$x_{i+1} = \mu x_i(1 - x_i)$$

$$x_{i+1} = f(x_i)$$ General notation for a map.

In certain cases, $f(x^*) = x^*$ x* is a "fixed point."

$$x^* = \mu x^*(1 - x^*)$$

$x^* = \frac{\mu - 1}{\mu}$ or $x^* = 0$ x* can be stable or unstable

Suppose you are close to but not exactly at x*

Do you head toward x* or …. away?

$$x_n = x^* + \eta_n$$ ⬆ or ⬇

$$x^* + \eta_{n+1} = x_{n+1} = f(x^* + \eta_n) = \underbrace{f(x^*) + f'(x^*)\eta_n + \mathcal{O}(\eta_n^2)}_{\text{Mr. Taylor ….}}$$

$$\eta_{n+1} = f'(x^*)\eta_n + \mathcal{O}(\eta_n^2)$$

Fixed points and 1D maps (2)

$$\eta_{n+1} = f'(x^*)\eta_n + \mathcal{O}(\eta_n^2)$$

$$\eta_{n+1} \simeq f'(x^*)\eta_n$$

Note that if $|f'(x^*)| < 1$, $\eta_{n+1} \to 0$. x* is a stable fixed point.

Note that if $|f'(x^*)| > 1$, $\eta_{n+1} \to \infty$. x* is an unstable fixed point.

$$\frac{df}{dx}|_{x^*} = \mu - 2\mu x^* = \begin{cases} \mu \\ 2 - \mu \end{cases}$$

Stable at $x^* = 0$ if $\mu < 1$

Stable at $x^* = (\mu - 1)/\mu$ if $\mu < 3$

Summary

How to use multiple parameters in an ODE. (vdpol.cc)

C++ `struct` structure

Intro to the logistic map

Don't suffer in silence. Scream for help!!!

Appendix

Orbit slides for easy reference

Reminder: Simultaneous 2nd-order ODEs

Suppose we have a pair of simultaneous 2nd-order ODEs.

$$\ddot{x} = f(t, x, y, \dot{x}, \dot{y})$$

$$\ddot{y} = g(t, x, y, \dot{x}, \dot{y})$$

What do we do?

$$d\vec{y}/dt = \vec{f}(t, \vec{y})$$ ("standard "form")

$$\vec{y} = \begin{pmatrix} y^{(0)}(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(N-1)}(t) \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} f^{(0)}(t, \vec{y}) \\ f^{(1)}(t, \vec{y}) \\ \vdots \\ f^{(N-1)}(t, \vec{y}) \end{pmatrix}$$

$$y^{(0)}(t) = x(t) \qquad y^{(2)}(t) = y(t)$$

$$y^{(1)}(t) = \dot{x}(t) \qquad y^{(3)}(t) = \dot{y}(t)$$

Repeat: Example of Simultaneous 2nd-order ODEs

What to use for $f^{(0)}$, $f^{(1)}$, …?

$$f^{(0)}(t,\vec{y}) = y^{(1)}(t) \quad (= \dot{x})$$

$$f^{(1)}(t,\vec{y}) = \dot{y}^{(1)}(t) \quad (= \ddot{x})$$

$$f^{(2)}(t,\vec{y}) = y^{(3)}(t) \quad (= \dot{y})$$

$$f^{(3)}(t,\vec{y}) = \dot{y}^{(3)}(t) \quad (= \ddot{y})$$

If we had more coupled equations, we just add pairs of y and f as here.

Example: Planetary motion. (See *CP*, sec 15.11)

$$f = -\frac{GMm}{r^2}$$

Attractive force exerted on m by M along center-line.

Take to be the sun

$$\vec{f} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

$$f_x = f\cos\theta = f\,\frac{x}{r} \qquad f_y = f\sin\theta = f\,\frac{y}{r} \qquad r = \sqrt{x^2+y^2}$$

$$\boxed{\frac{d^2x}{dt^2} = -GM\frac{x}{r^3}} \qquad \boxed{\frac{d^2y}{dt^2} = -GM\frac{y}{r^3}}$$

Reminder: Elements of Orbital Mechanics

Consider the case where a small object (e.g., a comet) orbits the Sun.

$$\vec{F} = -\frac{GMm}{|r|^2}\hat{r}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\vec{L} = \vec{r} \times (m\vec{v})$$

conserved

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{GM/r}$$

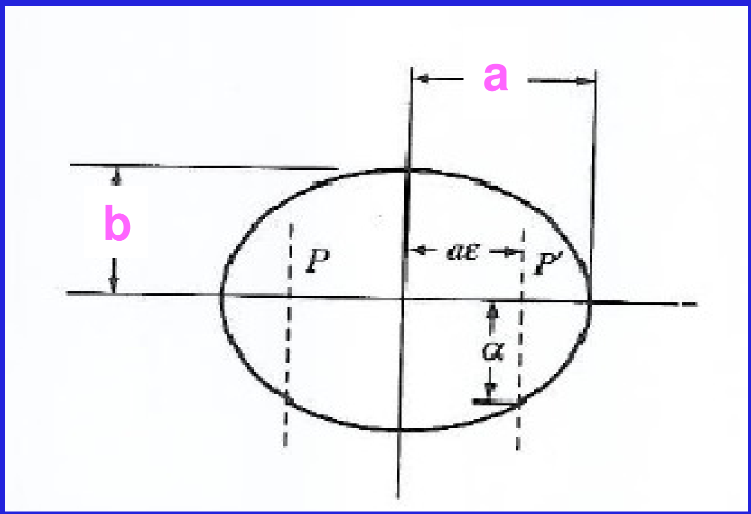
$$E = -\frac{GMm}{2r}$$

Circular orbits only

$$e = \sqrt{1 - b^2/a^2}$$

a = “semi-major axis”

b = “semi-minor axis”



Earth_e = 0.017

Elliptical Orbits and Kepler's 3rd Law

For <u>elliptical</u> orbits around Sun, we have:

$$E = -\frac{GMm}{2a}$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}$$

Exact only for m << M

$$G = 6.67 \times 10^{-11}\ \mathrm{m^3/kg{-}sec^2}$$

$$1\ \mathrm{AU} = 150 \times 10^6\ \mathrm{km} = 1.50 \times 10^{11}\ \mathrm{m}$$

$$1\ \mathrm{yr} = \pi \times 10^7\ \mathrm{sec}$$

$$M_\odot = 2.0 \times 10^{30}\ \mathrm{kg}$$

[r] = AUs

[t] = years

$$GM_\odot = 4\pi^2\, AU^3/yr^2$$

Useful for calc's w/ orbit.cc

orbit.cc

In class exercise:

Copy orbit.cc

For simplicity, set GM = 1.

Initial conditions:

$x(0) = 0.5$ $y(0) = 0.0$ $v_x(0) = 0.0$ $v_y(0) = 1.63$

Adjust number and size of time steps to see orbit close.

Visualize w/ `gnuplot`.

Q: Suppose $f = kr^{-n},\ n \neq 2$ How is orbit effected?

Mercury's orbit shows such an effect. $f = kr^{-2} + k'r^{-4}$

Gen'l Relativity