Lecture 13 Review

How to use multiple parameters in an ODE. (vdpol.cc)

C++ struct structure

Intro to the logistic map $x_{i+1} = \mu x_i (1 - x_i)$
Chaotic Pendulum

Examine phase space plots \((dx/dt \text{ v. } x)\) for \textit{driven}, damped pendulum.

\[ \tau_g + \tau_f + \tau_{ext} = I \frac{d^2 \theta}{dt^2} \]

\[ -\frac{mgl}{I} \sin \theta - \frac{\beta}{I} \frac{d\theta}{dt} + \frac{\tau_0}{I} \cos \omega t = \frac{d^2 \theta}{dt^2} \]

\[ \frac{d^2 \theta}{dt^2} = -\omega_0^2 \sin \theta - \alpha \frac{d\theta}{dt} + f \cos \omega t \]

Verify oscillation of free, undamped pendulum.

Plot \(\theta(t)\) v. \(t\);

\(\text{Choose } \omega_0^2 = 1 \text{ rad/sec}^2\)

Plot \(d\theta/dt\) v. \(\theta\).

Similar to HW.
Examine phase space plots \((dx/dt \text{ v. } x)\) for driven pendulum

\[
\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin \theta - \alpha \frac{d\theta}{dt} + f \cos \omega t
\]

- Consider damped pendulum \((f = 0.)\) Plot \(\theta(t)\) v. \(t\); Plot \(d\theta/dt\) v. \(\theta\).
  - Plot \(\theta(t)\) v. \(t\); Plot \(d\theta/dt\) v. \(\theta\).
  - Choose \(\omega_0^2 = 1\) \(\text{rad/sec}^2\)
  - Choose \(\alpha = 0.2 /\text{sec}\)

- Consider damped and driven pendulum.
  - Plot \(\theta(t)\) v. \(t\); Plot \(d\theta/dt\) v. \(\theta\).
  - Choose \(\omega_0^2 = 1\)
  - Choose \(\alpha = 0.2\)
  - Choose \(\omega = 0.666\)
  - \((x_0, v_0) = (-0.0885, 0.8)\)
  - \((x_0, v_0) = (-0.0883, 0.8)\)
  - \((x_0, v_0) = (-0.0888, 0.8)\)
Chaotic Pendulum (3)
Chaotic motion is motion w/o any apparent regularity.

Chaotic motion is NOT random motion.

Random motion means you cannot predict future motion from present, even in principle.

- However, relevant chaotic ODE tells you how to get from present to future.
- IF you start w/ identical ICs, you always get the same final state.
- It is extreme sensitivity of a chaotic ODE to initial conditions that makes practical prediction of far future motion impossible.
- Change ICs slightly for chaotic system, very different final state.

Consider the example of the logistic map.
Chaos Identification (2)

\[ x_{N+1} = \alpha x_N (1 - x_N) \]

Set \( \alpha = 4.0 \)

Pick \( x_1 = 0.700\,000\,000 \)

\[ = 0.700\,000\,001 \]

two different ICs (float v. double \ldots)

Find iteration \( N \) where the 2 solutions have clearly diverged.

\( N = ?? \)
Suppose difference between 2 sols, $\Delta$, **doubles** every iteration.

After $N$ iterations:

$$\Delta = 2^N = e^{N \ln 2}$$

For final $\Delta \sim 1$:

$$2^N 10^{-8} \sim 1$$

$$\Rightarrow N = 27$$

Consider 2 initial states:

- $x_0$
- $x_0 + \varepsilon \quad \varepsilon << 1$

Lyapunov exponent.

**Exponential growth in solution difference $\Delta$ if $\lambda > 0$.**
Let’s try to make it. (Actually not so hard.)

See sierpin.cc
Sierpinski’s Gasket Algorithm

- Draw equilateral triangle and label vertices (1,2,3).
- **Randomly** pick a single point $P_0$ inside triangle.
- **Randomly** pick an integer from \{1,2,3\}.
- Place 2\textsuperscript{nd} point halfway between $P_0$ and vertex from previous step.
- Call this new point $P_0$ and repeat last 3 steps.

\[
(x_{k+1}, y_{k+1}) = \frac{(x_k, y_k) + (Vx_n, Vy_n)}{2}
\]

\[
n = \text{integer} (1 + 3r_i)
\]

See sierpin.cc
Fractal: “shape made of parts similar to the whole”

**Typical fractal $F$ properties:**

- $F$ has structure at arbitrarily small scales.
- $F$ is self-similar. (NB: Not all fractals self-similar.)
- $F$ has non-integral dimension (say what?).

Sierpinski gasket has all 3 properties.
Some Other (Self-Similar) Fractals

Cantor Set:

Koch Curve:
Q: What do you mean by “dimension”?

- Maybe, number of numbers required to specify location of a point?
- Works OK for line segment and planar shapes. So far, so good.
- Concept fails spectacularly for Koch curve K.

K has infinite arc length!

\[
\begin{align*}
L_0 & \sim S_0 \\
L_1 & \sim S_1 \\
L_1 &= \frac{4}{3}L_0 \\
L_2 &= \frac{4}{3}L_1 = \left(\frac{4}{3}\right)^2L_0 \\
&\vdots \\
L_n &= \left(\frac{4}{3}\right)^nL_0 \to \infty \text{ as } n \to \infty
\end{align*}
\]
Q: What do you mean by “dimension”?
Examine simple, self-similar structures.

\[ m = r^d \]
\[ d = \frac{\ln m}{\ln r} \]

“similarity” dimension.
Apply this definition of dimension to $K$.

Compare $S_2$ with $S_1$:
- $S_1$ reduced by a factor of 3.
- 4 such identical segments span $S_1$.

$$r = 3, \quad m = 4$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 4}{\ln 3} = 1.26$$
Apply our definition of dimension.

\[ d = \frac{\ln m}{\ln r} \]

Compare \( S_2 \) with \( S_1 \):
- \( S_1 \) reduced by a factor of 3.
- 2 such identical segments span \( S_1 \).

\[ \Rightarrow r = 3 \]
\[ m = 2 \]

\[ d = \frac{\ln 2}{\ln 3} = 0.63 \]
Summary

Chaos Intro w/ driven, damped pendulum.
Chaos ID and Lyapunov exponent
First look at simple fractals
Fractal dimension

Don’t suffer in silence. Scream for help!!!