Lecture 13 Review

How to use multiple parameters in an ODE. (vdpol.cc)

C++ `struct` structure

Intro to the logistic map $\boxed{x_{i+1} = \mu x_i(1 - x_i)}$

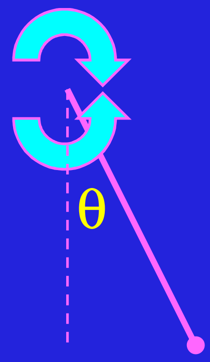
Chaotic Pendulum

Examine phase space plots (dx/dt v. x) for driven, damped pendulum.

$$\tau_g + \tau_f + \tau_{ext} = I\frac{d^2\theta}{dt^2}$$

$$-\frac{mgl}{I}\sin\theta - \frac{\beta}{I}\frac{d\theta}{dt} + \frac{\tau_0}{I}\cos\omega t = \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\sin\theta - \alpha\frac{d\theta}{dt} + f\cos\omega t$$



Verify oscillation of free, undamped pendulum.

Plot $\theta(t)$ v. t;

Plot $d\theta/dt$ v. θ.

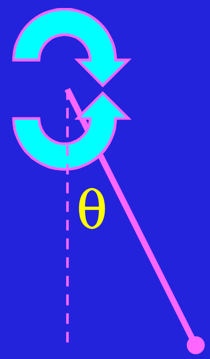
Choose $\omega_0^2 = 1$ rad/sec^2

Similar to HW.

Chaotic Pendulum (2)

Examine phase space plots (dx/dt v. x) for driven pendulum

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta - \alpha\frac{d\theta}{dt} + f\cos\omega t$$



- Consider damped pendulum (f =0.) Plot θ(t) v. t; Plot dθ/dt v. θ.

  Plot θ(t) v. t; Plot dθ/dt v. θ.

  Choose ω_0^2 = 1 rad/sec^2

  Choose α = 0.2 /sec

- Consider damped and driven pendulum.

  Plot θ(t) v. t; Plot dθ/dt v. θ.

  Choose ω_0^2 = 1   Choose f = 0.52

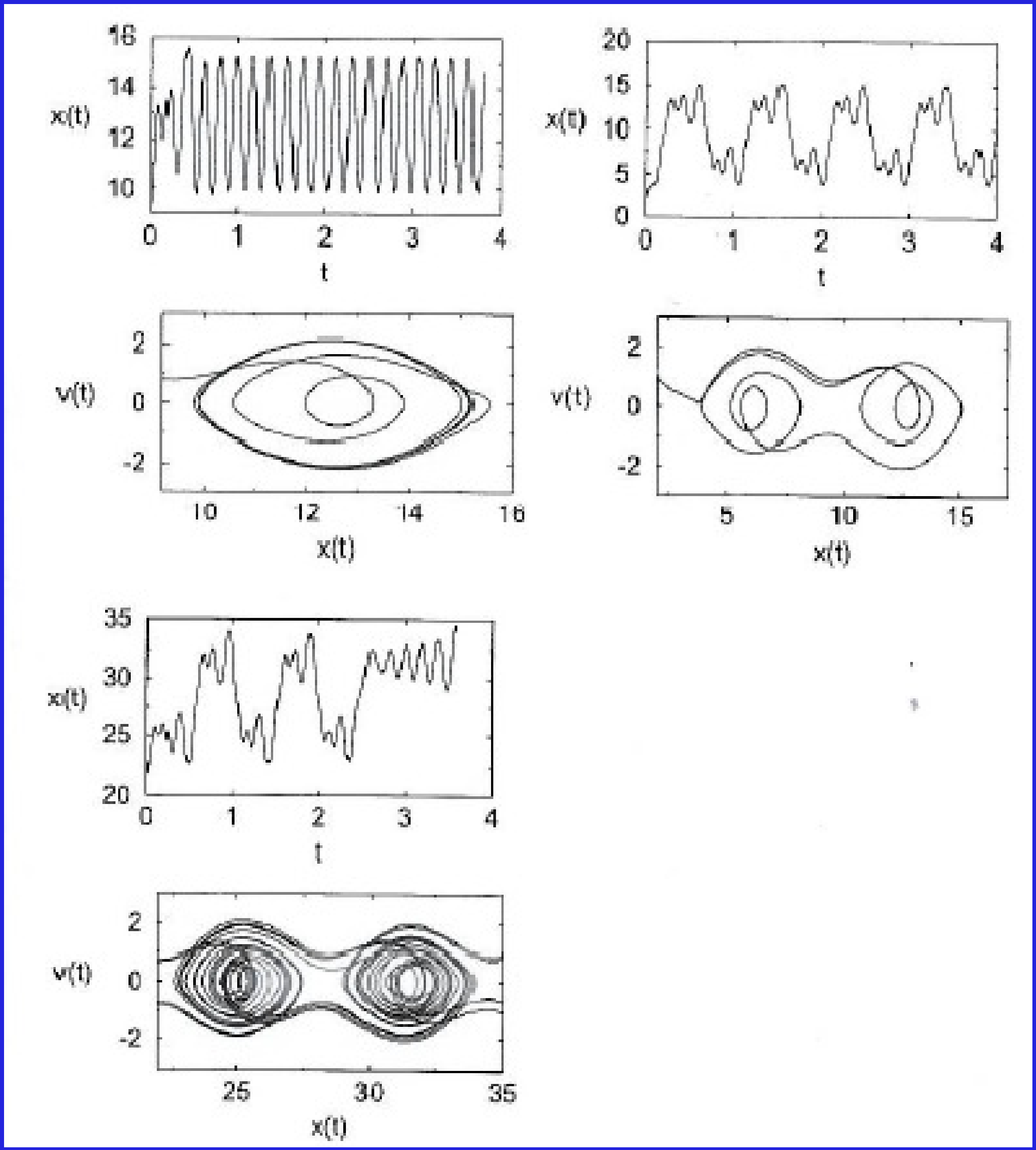
  Choose α = 0.2   Choose ω = 0.666

(x_0, v_0) = (-0.0885,0.8)

(x_0, v_0) = (-0.0883,0.8)

(x_0, v_0) = (-0.0888,0.8)

Chaotic Pendulum (3)



Chaos Identification (Qualitative)

- *Chaotic* motion is motion w/o any apparent regularity.
- Chaotic motion is NOT random motion.
- Random motion means you cannot predict future motion from present, even in principle.

➢ However, relevant <u>chaotic</u> ODE tells you how to get from present to future.

➢ IF you start w/ <u>identical</u> ICs, you <u>always</u> get the same final state.

➢ It is extreme sensitivity of a chaotic ODE to initial conditions that makes <u>practical</u> prediction of <u>far</u> future motion impossible.

➢ Change ICs slightly for chaotic system, very different final state.

Consider the example of the logistic map.

Chaos Identification (2)

$$x_{N+1} = \alpha x_N(1 - x_N)$$

Set $\alpha = 4.0$

Pick $x_1 = 0.700\ 000\ 000$

$= 0.700\ 000\ 001$

two different ICs

(float v. double …)

Find iteration N where the 2 solutions have clearly diverged.

N = ??

Chaos Identification (3)

$$x_{N+1} = \alpha x_N (1 - x_N)$$

Suppose difference between 2 sols, Δ, doubles every iteration.

After N iterations: $\Delta = 2^N = e^{N \ln 2}$

For final $\Delta \sim 1$: $2^N \, 10^{-8} \sim 1$

$\Rightarrow N = 27$

Starting difference between ICs

Consider 2 initial states: x_0

$x_0 + \varepsilon \qquad \varepsilon << 1$

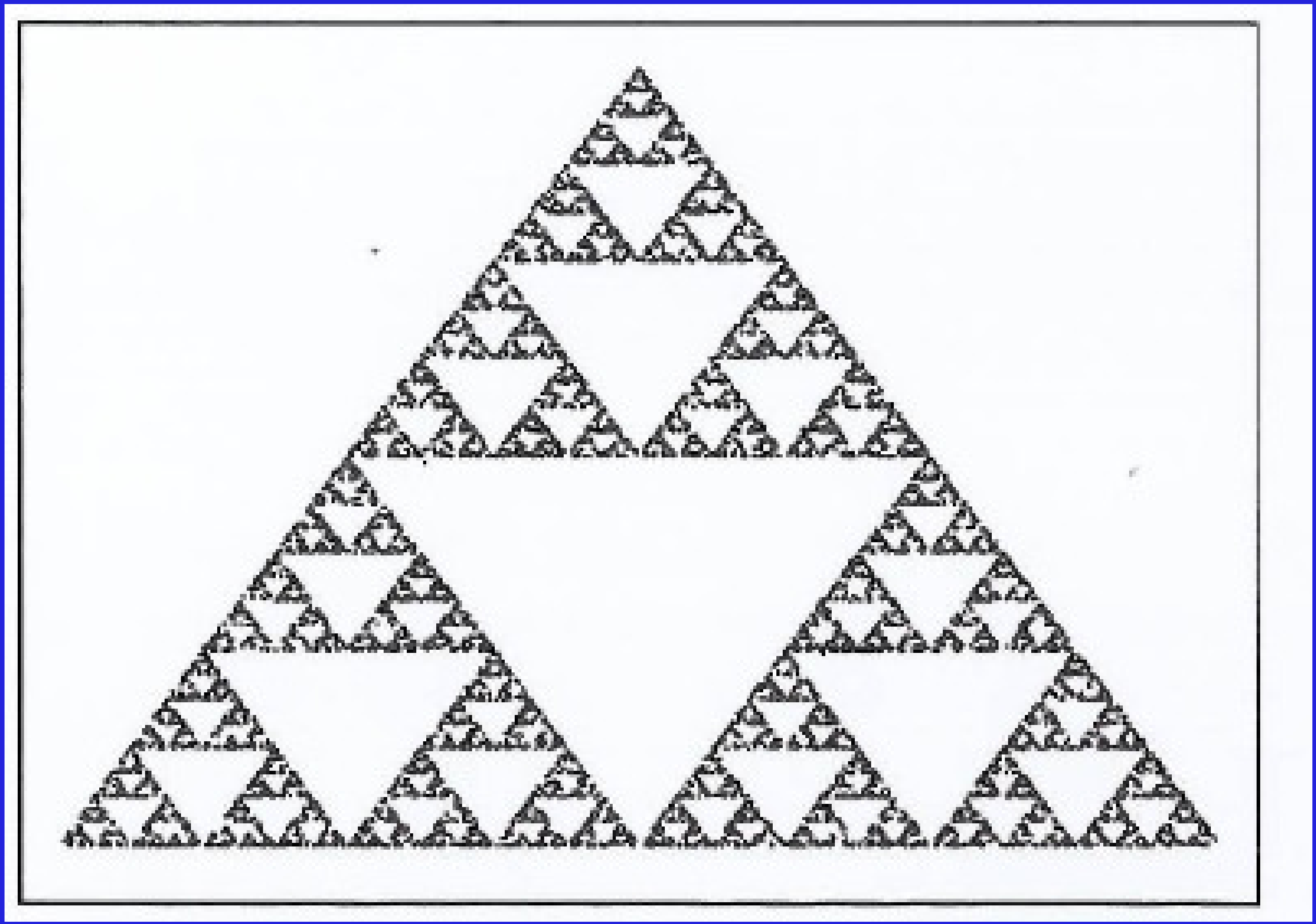
$$\Delta_N = x_N^{(1)} - x_N^{(2)}$$

$$\Delta_N = \epsilon e^{N\lambda}$$

Lyapunov exponent.

Exponential growth in solution difference Δ if $\lambda > 0$.

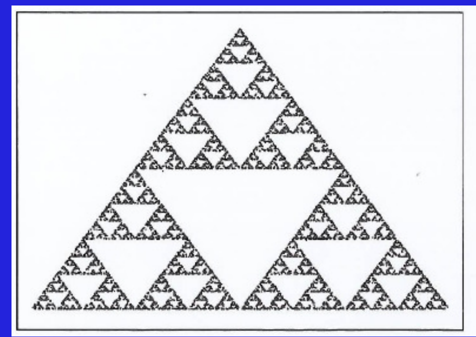
Fractals (Play Time)



Let's try to make it. (Actually not so hard.)

See sierpin.cc

Sierpinski's Gasket Algorithm



- Draw equilateral triangle and label vertices (1,2,3).
- Randomly pick a single point P_0 inside triangle.

N

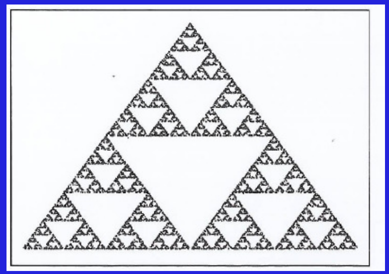
- Randomly pick an integer from {1,2,3}.
- Place 2nd point halfway between P_0 and vertex from previous step.
- Call this new point P_0 and repeat last 3 steps.

$$(x_{k+1}, y_{k+1}) = \frac{(x_k, y_k) + (Vx_n, Vy_n)}{2}$$

$$n = \text{integer}\,(1 + 3r_i)$$

See sierpin.cc

Typical Fractal Properties



Fractal: “shape made of parts similar to the whole”

Typical fractal *F* properties:

F has structure at arbitrarily small scales.

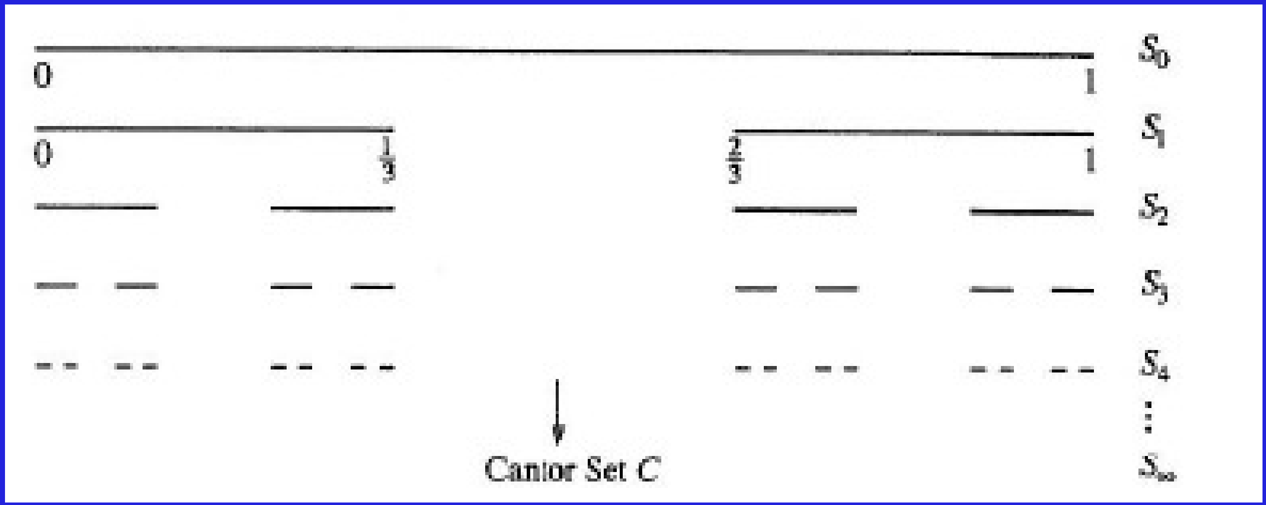
F is self-similar. (NB: Not all fractals self-similar.)

F has non-integral dimension (say what?).

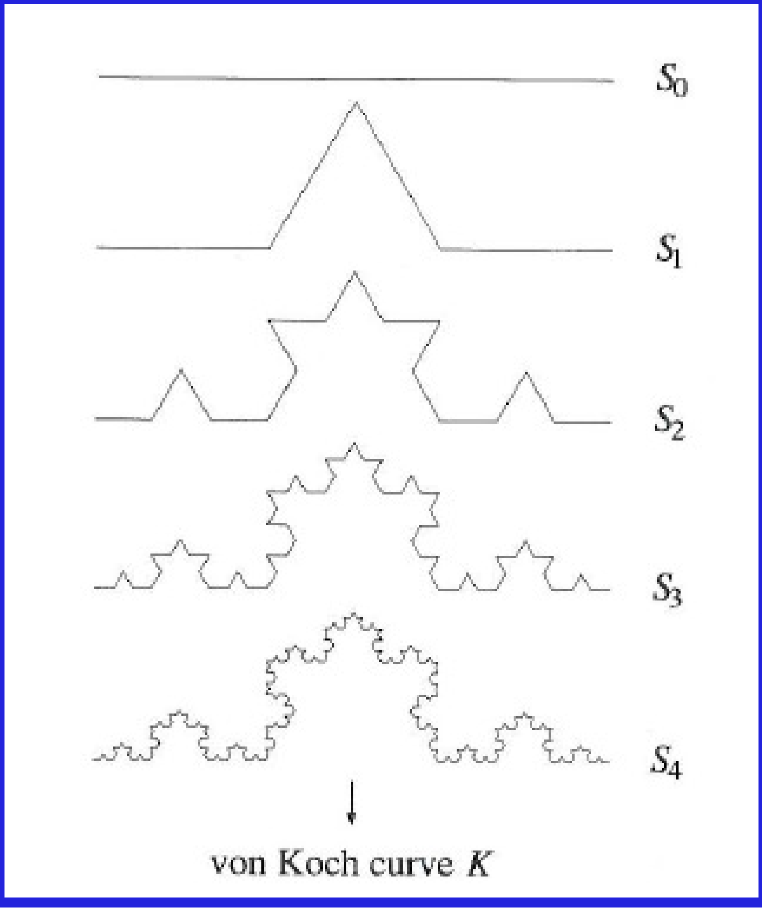
Sierpinski gasket has all 3 properties.

Some Other (Self-Similar) Fractals

Cantor Set:



Koch Curve:



Dimension of Self-Similar Fractals

Q: What do you mean by “dimension”?

- Maybe, number of numbers required to specify location of a point?
- Works OK for line segment and planar shapes. So far, so good.
- Concept fails spectacularly for Koch curve K.

K has infinite arc length!

$$L_0 \sim S_0$$

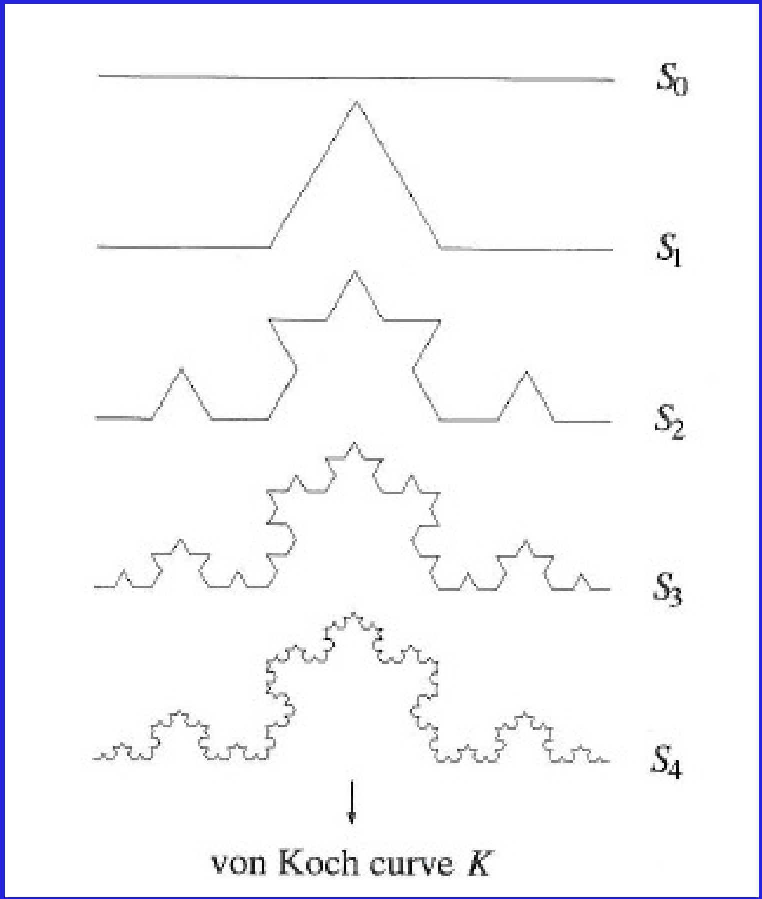
$$L_1 \sim S_1$$

$$L_1 = \frac{4}{3} L_0$$

$$L_2 = \frac{4}{3} L_1 = (\frac{4}{3})^2 L_0$$

$$\vdots$$

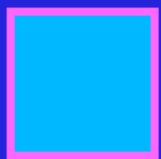
$$L_n = (\frac{4}{3})^n L_0 \to \infty \text{ as } n \to \infty$$

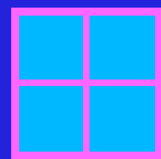


Dimension of Self-Similar Fractals (2)

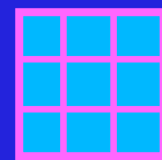
Q: What do you mean by “dimension”?

Examine simple, self-similar structures.

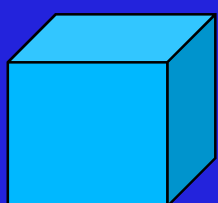


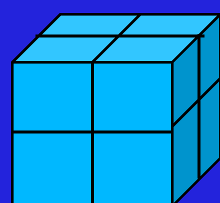


$m = 4$
$r = 2$



$m = 9$
$r = 3$





$m = 8$
$r = 2$

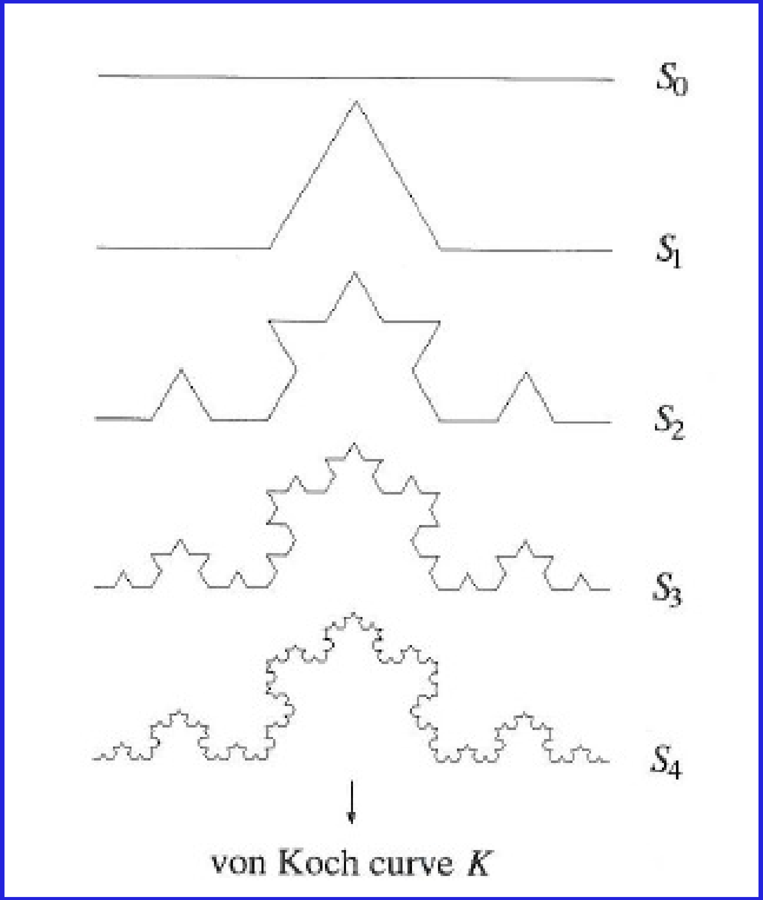
$$\boxed{\begin{gathered} m = r^d \\ d = \frac{\ln m}{\ln r} \end{gathered}}$$

“similarity” dimension.

Dimension of Koch Curve K

Apply this definition of dimension to K.

$$d = \frac{\ln m}{\ln r}$$



Compare S_2 with S_1:

- S_1 reduced by a factor of 3.
- 4 such identical segments span S_1.

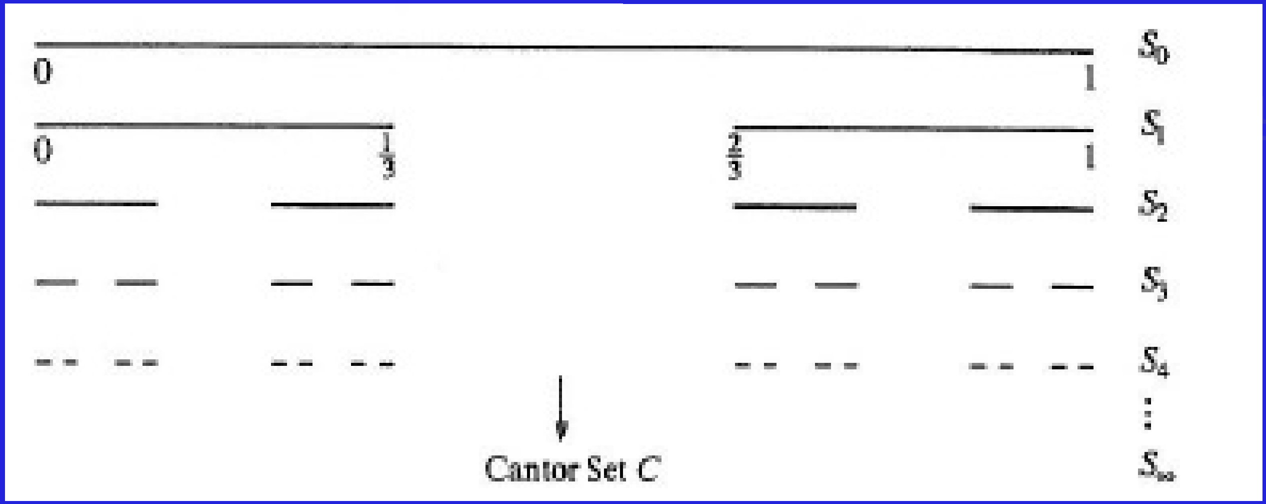
$$\Rightarrow \begin{aligned} r &= 3 \\ m &= 4 \end{aligned}$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 4}{\ln 3} = 1.26$$

Dimension of Cantor Set C

Apply our definition of dimension.

$$d = \frac{\ln m}{\ln r}$$



Compare S_2 with S_1:

- S_1 reduced by a factor of 3.
- 2 such identical segments span S_1.

$$\Rightarrow \begin{aligned} r &= 3 \\ m &= 2 \end{aligned}$$

$$d = \frac{\ln m}{\ln r} = \frac{\ln 2}{\ln 3} = 0.63$$

Summary

Chaos Intro w/ driven, damped pendulum.

Chaos ID and Lyapunov exponent

First look at simple fractals

Fractal dimension

Don't suffer in silence. Scream for help!!!