Lecture 16 Review

Newton-Raphson warnings.

`numerov.cc` discussion.

Solution of triangular well via `numerov.cc`

Fourier Series

Fourier Decomposition

We often observe periodic phenomena in Nature.

Period

Describe by periodic functions: y(t + T) = y(t).

y(t) could look very complicated in general.

Easier to think of y(t) as a superposition of simpler functions.

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

numbers

Not obvious this is true.

E.g., How do you know the infinite series even converges?

Fourier Series (2)

Well, Dirichlet's Theorem to the rescue.

- If y(t) periodic w/ period 2π
- If y(t) has finite # of discontinuities AND finite #'s of max's and min's all for $-\pi < t < \pi$
- If $\int_{-\pi}^{\pi} f(t)\, dt$ = finite

THEN FS converges to f(t), where f(t) is continuous.

At jump points, converges to arithmetic mean of of LH & RH limits of f(t).

Fourier Series (3)

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$$

w/ $\omega T = 2\pi$

How to determine coefficients a_n and b_n ?

$$a_n = \frac{2}{T}\int_0^T dt\ \cos n\omega t\, y(t)$$

$$b_n = \frac{2}{T}\int_0^T dt\ \sin n\omega t\, y(t)$$

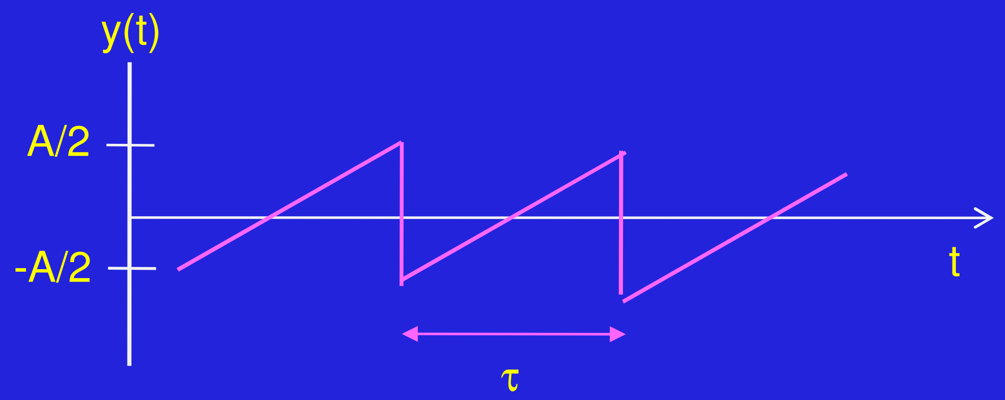
If y(t) is <u>ODD</u>, $y(-t) = -y(t)$, $a_n = 0$, $\forall n$

If y(t) is <u>EVEN</u>, $y(-t) = y(t)$, $b_n = 0$, $\forall n$

Fourier Series (4)

Example

Sawtooth Function



$$y(t) = At/\tau = \frac{\omega A}{2\pi}t, \quad -\tau/2 < t < \tau/2$$

y(t) = odd $\Rightarrow$ $a_n = 0 \;\; \forall n.$

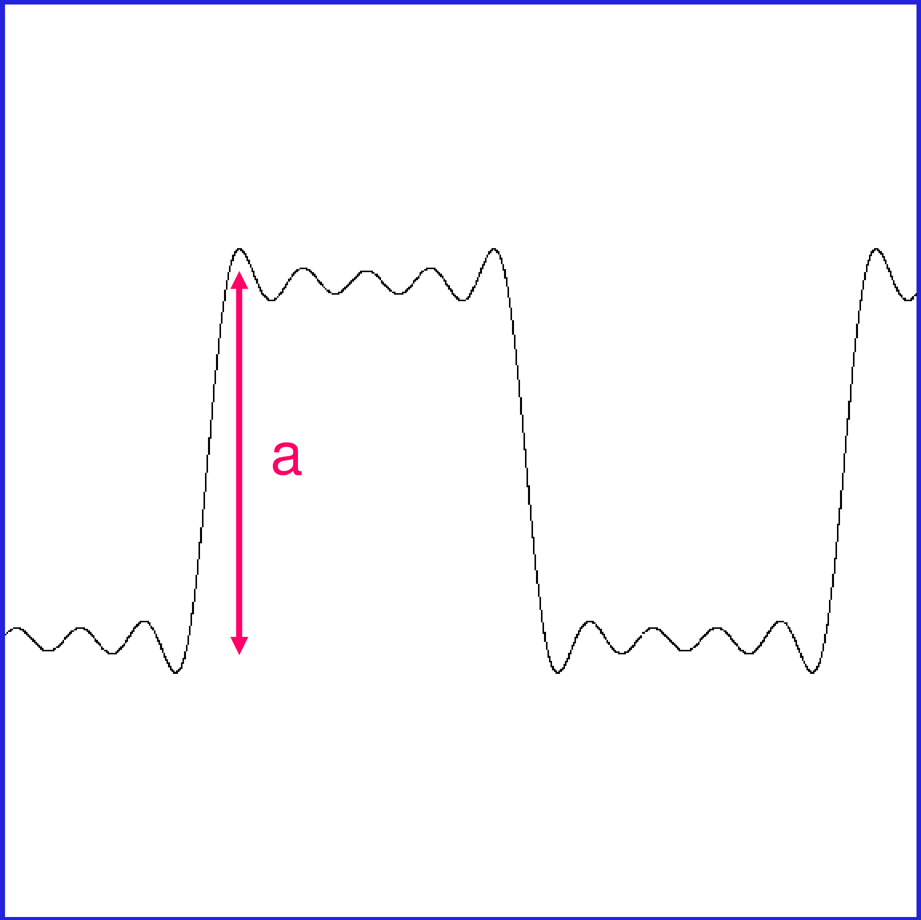
$$\begin{aligned} b_n &= \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \,\sin n\omega t\, At/\tau \\ &= \frac{\omega^2 A}{2\pi^2}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} dt\, t \sin n\omega t \\ &= \frac{\omega^2 A}{2\pi^2}\left[-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right]\Big|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} \end{aligned}$$

Fourier Series (5)

$$
\begin{aligned}
b_n &= \frac{\omega^2 A}{2\pi^2}\left(\frac{2\pi}{n\omega^2}\right)(-1)^{n+1} \\
&= \frac{A}{n\pi}(-1)^{n+1}
\end{aligned}
$$

Lab exercise: verify w/ gnuplot. Use 5 terms.

Fourier Series (6)



- Gibbs phenomenon (9% overshoot as n $\rightarrow \infty$).
  Try sawtooth w/ 8 terms.

- At jump, FS = arithmetic mean(LHS & RHS).

Fourier Transforms

Fourier series good for periodic functions.

Fourier “transforms” good for non-periodic functions.

FT → $H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$

Inverse FT → $h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$

our choice

FT → $H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{i\omega t}\, dt$

$\omega = 2\pi f$

Inverse FT → $h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega f t}\, d\omega$

H(f) is a measure of the contribution of a particular frequency to f (t) .

Fourier Transforms (2)

We can think of ‘time space” and “frequency space.”

h(t) and H(f) have various symmetry properties:

h(t) even	H(f) even
h(t) odd	H(f) odd
h(t) real	H(-f) = [H(f)]
h(t) imaginary	H(-f) = - [H(f)]*

Parseval’s Theorem:

$$\text{''total power''} \equiv \int_{-\infty}^{\infty} |h(t)|^2\, dt = \int_{-\infty}^{\infty} |H(f)|^2\, df$$

“one-sided power spectral density” (PSD)

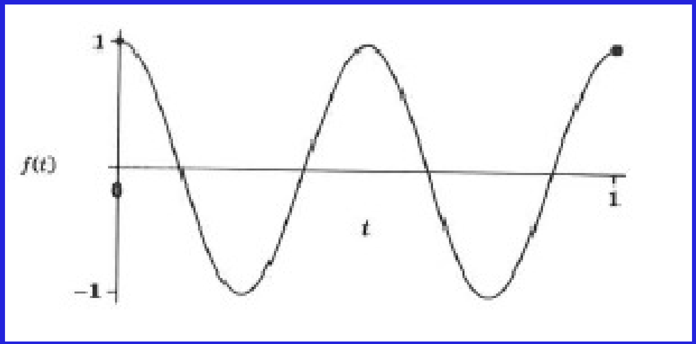
$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \quad 0 \leq f < \infty$$

Discrete Fourier Transform

Issue: Often discretely sample a waveform and need to know:

its shape and/or its frequency characteristics.

Sample (i.e., measure w/ an instrument) every Δ seconds.
(1/Δ is the "sampling rate")



cos(2πf t)   f = 2
sampled every 1 second

"Nyquist critical frequency" $f_c \equiv \frac{1}{2\Delta}$

Discrete Fourier Transform (2)

Let

$h_k \equiv h(t_k)$ $\quad t_k \equiv k\Delta$ $\quad k = 0, 1, 2 \ldots, N-1$

$f_n \equiv \frac{n}{N\Delta}$ $\quad n = -\frac{N}{2}, \ \ldots \ \frac{N}{2}$

N = number of data points

Δ = time between data points.

$f_n < 0$? Looks weird. More on this shortly.

Discretize integral form of FT:

$$H(f_n) = \int_{-\infty}^{\infty} h(t)e^{2\pi i f_n t}\,dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k}\Delta = \Delta \underbrace{\sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}}_{\text{DFT}}$$

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}$$ DFT

same units

$H(f_n) \approx \Delta H_n$ continuous v. discrete FTs (note the units)

$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}$ Inverse DFT

$\sum_{k=0}^{N-1} |h_k|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |H_n|^2$ Discrete form of Parseval's theorem

DFT (3)

$$\boxed{H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}}$$

- Note: H_n has periodicity of N. H(n) = H(N-n)*

$\rightarrow$ Prove this statement ! $\leftarrow$

You do <u>not</u> have 2*N independent numbers in H_n ! <u>Only</u> N.

Same as number of sampled points h_k

(You will see this in DFT data file.)

Since H(-n) = H(N-n), n = 1,2, …

The n index in H_n varies from 0,1, …N-1 (same as k index)

$\longrightarrow$ $\boxed{f_n \equiv \frac{n}{N\Delta}}$ w/ index n now varying from 0,1, …N-1

```
make_fourier_data.cc
dft.cc
inv_dft.cc
```

DFT(4)

All in one place:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t}\, df$$

Our choice.

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{i\omega t}\, dt$$

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega f t}\, df$$

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n / N}$$

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n / N}$$

$$f_n \equiv \frac{n}{N\Delta}$$

n = 0,1, 2 … N - 1.

N = # of sampled data points.

Δ = time between sampled data points.

Δ^{-1} = sampling rate.

Summary

Fourier Series.

DFT (theory and lab example).

Don’t suffer in silence. Scream for help!!!