Lecture 18 Review

Sampling theory and aliasing.

Nyquist critical frequency

FFT fun (theory and exercises)

(pick N = power of 2)
Physics often requires solution of simultaneous linear equations.
e.g., coupled oscillators, electrical circuits, …

Set of equations of the form: \( Ax = b \) Solve for \( x, A \) & \( b \) given.

Physics often requires solution of eigenvectors & eigenvalues.
e.g., normal modes, eigenfrequencies, bound energy states, …

Equations of the form: \( Ax = \lambda x \) Solve for \( x, \lambda \) & \( A \) given.

- These are 2 different classes of problems to solve.
- Techniques are sophisticated. We will use canned software.
Solution of Linear Simultaneous Equations

Gaussian elimination. Easiest to understand.

\[
\begin{align*}
2u + v + w &= 1 \\
4u + v &= -2 \\
-2u + 2v + w &= 7 \\
2u + v + w &= 1 \\
-1v - 2w &= -4 \\
3v + 2w &= 8 \\
2u + v + w &= 1 \\
-1v - 2w &= -4 \\
-4w &= -4
\end{align*}
\]

Count number of operations

- “Forward elimination.”
- “Back substitution.”

Where might this technique break down?
LU Decomposition

Write matrix $A = LU$ i.e., factorize $A$, always OK if $A$ has non-zero pivots

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{=} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_{10} & 1 & 0 & 0 \\ \alpha_{20} & \alpha_{21} & 1 & 0 \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} \\ 0 & \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & \beta_{33} \end{bmatrix}$$

$Ax = b = (LU)x = L(Ux) = b$

$Ly = b$

$Ux = y$

a way to proceed.

example:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix}$$

or

$$c = \begin{bmatrix} 8 \\ -5 \\ -4 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -4 \end{bmatrix}$$

or

$$x = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$
LU Decomposition (2)

\[
\begin{bmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} \\
  a_{10} & a_{11} & a_{12} & a_{13} \\
  a_{20} & a_{21} & a_{22} & a_{23} \\
  a_{30} & a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

= \[
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  \alpha_{10} & 1 & 0 & 0 \\
  \alpha_{20} & \alpha_{21} & 1 & 0 \\
  \alpha_{30} & \alpha_{31} & \alpha_{32} & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} \\
  0 & \beta_{11} & \beta_{12} & \beta_{13} \\
  0 & 0 & \beta_{22} & \beta_{23} \\
  0 & 0 & 0 & \beta_{33}
\end{bmatrix}
\]

Ax = b = (LU)x = L(Ux) = b
Ly = b
Ux = y

{a way to proceed.}

\[
y_0 = \frac{b_0}{\alpha_{00}}
\]

“forward substitution”

\[
y_i = \frac{1}{\alpha_{ii}} \left[ b_i - \sum_{j=0}^{i-1} \alpha_{ij} y_j \right] \quad i = 1, 2, \ldots, N - 1
\]

“back substitution”

\[
x_{N-1} = \frac{y_{N-1}}{\beta_{N-1,N-1}}
\]

L & U computed once per A

N^3 steps to solve for x.
What to do if A has zero pivots?
If A has an inverse (i.e., is "non-singular"), reorder rows of A beforehand to prevent zero pivots $A \rightarrow PA$

$PA = LU$

$P = "permutation matrix"$ (reorders rows of A)

$$P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

e.g., swaps row 2 & 4 of A

$PAx = Pb$ has same solution $x$ as $Ax = b$. (Formally, $x = A^{-1}b$)

Proof: $x = (PA)^{-1}Pb = A^{-1}P^{-1}Pb = A^{-1}b$

Q: If $Ax = b$, why not just compute $x = A^{-1}b$?

A: Computing $A^{-1}$ is more “expensive” than computing LU.
Exercise (from chemistry !): \[ \alpha \text{O}_2 + \beta \text{C}_4\text{H}_9\text{NH}_2 \rightarrow \gamma \text{CO}_2 + \delta \text{H}_2\text{O} + \text{N}_2 \]
Find correct stoichiometry (i.e., find \( \alpha, \beta, \gamma, \delta \)).

Often need to perform matrix calculations quickly (i.e., w/o writing code)
Use \textit{octave} (freeware)

```
prompt> octave
octave:1>
```

Use as calculator: \( \frac{2}{83} \)

Standard set of math functions:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>cos</td>
<td>cosine</td>
</tr>
<tr>
<td>exp</td>
<td>exponential</td>
</tr>
<tr>
<td>log</td>
<td>Natural log</td>
</tr>
<tr>
<td>log10</td>
<td>Log base 10</td>
</tr>
<tr>
<td>tanh</td>
<td>Hyperbolic tangent</td>
</tr>
<tr>
<td>atan</td>
<td>Arc-tangent</td>
</tr>
<tr>
<td>round</td>
<td>Round to nearest integer</td>
</tr>
</tbody>
</table>

Colon notation:
```
octave:31>: e = 2:6
```
```
octave:31>: f = 2:6:40
```

Semicolon usage:
```
octave:31>: f = 2:6;
```
Extensive help utility: try \texttt{help \ -i}

Matrix manipulation

\texttt{octave:45> a= [ 1,3;2,7]}
\texttt{octave:46> a'}
\texttt{octave:47> f = [ 1:6]'}

Built-in functions for large matrices.

\texttt{octave:51> s = zeros(M,N) w/ M,N = integers}
\texttt{octave:52> r = ones(M,N)}
\texttt{octave:53> rr = linspace(x1,x2,N)}
\texttt{octave:53> r = logspace(x1,x2,N)}
Plotting: basic command is \texttt{plot(x,y)}
uses gnuplot

\begin{verbatim}
octave:85> angles = [0:pi/3:2*pi];
octave:87> y = sin(angles)
octave:88> plot (angles, y)
\end{verbatim}

\textbf{Functions:}
\begin{verbatim}
octave:151> function s = dub(x)
> s = 2*x;
> end
octave:152> dub(35)
\end{verbatim}
Ax = b

Define A in usual way.

\texttt{octave:3> a = [1,3,5; 1, 5, 6; 3, 7, 9]} for example.
\texttt{octave:4> b = [2, 5,9]’} for example.
\texttt{octave:4> x = a\b}

\texttt{a\b} is octave speak for \texttt{A^{-1}}.

octave does NOT compute the inverse of \texttt{a} to solve for \texttt{x}.

\textbf{Many} variants to LU decomposition.
These depend on structure of \texttt{A}: degree of symmetry, sparseness, …
Physics often requires solution of eigenvectors & eigenvalues.

e.g., normal modes, eigenfrequencies, bound energy states, …

Equations of the form: \( Ax = \lambda x \)

Solve for \( \lambda \) & \( x \), \( A \) is given.

Nonlinear equation.

Eigenvectors \( x \) lie in the “nullspace” of \( A - \lambda I \)

For \( \lambda \) to be an eigenvalue of \( A \):

1) non-zero \( x \) for which \( Ax = \lambda x \)

2) \( A - \lambda I \) is singular

3) \( \det(A - \lambda I) = 0 \)

Each is necessary and sufficient

#3 implies sum of n eigenvalues of \( A \) = sum of diagonal entries of \( A \)

\[
\begin{vmatrix}
(a_{11} - \lambda) & a_{12} & a_{13} \\
a_{21} & (a_{22} - \lambda) & a_{23} \\
a_{31} & a_{32} & (a_{33} - \lambda)
\end{vmatrix} = (a_{11} - \lambda)[(a_{22} - \lambda)(a_{33} - \lambda) - a_{32}a_{23}] + \cdots = 0
\]
Eigenvalues

#3 also implies:

If $A$ is triangular (lower or upper), $\lambda$’s appear on diagonal of $A$

\[
\begin{vmatrix}
(a_{11} - \lambda) & a_{12} & a_{13} \\
0 & (a_{22} - \lambda) & a_{23} \\
0 & 0 & (a_{33} - \lambda)
\end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0
\]

Now, suppose the $n \times n$ matrix $A$ has $n$ linearly independent eigenvectors

Then if you write them as the column vectors of a matrix $S$

$S^{-1}AS$ is diagonal w/ the $\lambda$’s of $A$ along the diagonal:

\[
S^{-1}AS = \Lambda = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
\]
Here’s why $S^{-1}AS = \Lambda$

$AS = A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$

$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}$

$AS = S\Lambda$ or $(S^{-1}AS) = \Lambda$

“similarity transformation”

Furthermore, for any non-singular matrix $M$:
If $B = M^{-1}AM$ then $A$ & $B$ have same $\lambda$’s w/ same multiplicities.

Here’s why:

$\det(B - \lambda I) = \det(M^{-1}AM - \lambda I) = \det(M^{-1}(A - \lambda I)M)$

$= \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I)$
Strategy for Finding $\lambda$ and $x$

Strategy to find $\lambda$'s and eigenvectors $x$ of $A$.

Perform similarity transformations to diagonalize $A$

$$P^{-1}AP = A'$$ \quad \text{Diagonal (A and A' have same eigenvalues)}

$$P \equiv P_1P_2P_3 \ldots \quad \text{P could be a product of many transformations.}$$

$$P^{-1}AP = \ldots P_3^{-1}P_2^{-1}(P_1^{-1}AP_1)P_2P_3 \ldots$$

Amazingly, we can get eigenvectors from $P$.

Suppose the $n$ eigenvectors $u_i$ of $A$ are linearly indpt and are a basis.

$$v_i \equiv P^{-1}u_i$$

$$A'v_i = (P^{-1}AP)(P^{-1}u_i) = P^{-1}Au_i = \lambda v_i$$

Only non-zero term (remember, $A'$ is diagonal)

$$v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now note: $Pv_i = P(P^{-1}u_i) = u_i$

Eigenvectors of $A = \text{columns of P}$!
Examples of P-type Transformations

$$P = \begin{bmatrix} 1 \\ \vdots \\ -\sin \theta & \ldots & \cos \theta \\ \vdots \\ \cos \theta & \ldots & \sin \theta \\ 1 \end{bmatrix}$$

Works on some elements of A
Seek to eliminate off-diagonal terms

Also sometimes useful to factorize $A$: $A = PQ$
then note: $QP = (P^{-1}A)P$

similarity transformation
Q: What is the sum $S$ of the eigenvalues?
Q: What are the eigenvalues? Solve first by hand, then by `octave`.
FYI: `octave:19> help -i eig` will be helpful!!
Q: What is the LU decomposition of $A$?
NB: `octave` computes $A = PLU$ w/ 1’s along diagonal of $L$
Q: What are the eigenvectors of $A$?
Q: What is $A^{-1}$? Verify this.
Find I in all legs.
Recall Volterra prey-predator equations

\[
\begin{align*}
\frac{dx}{dt} &= x(a - by) \\
\frac{dy}{dt} &= -y(c - dx)
\end{align*}
\]

Solved previously w/ GSL routines.
Can be solved w/ octave

```octave
function xdot = vp(x,t)
    xdot = zeros(2,1);
    a = 1.0;
    b = 0.5;
    c = 0.95;
    d = 0.25;
    xdot(1) = x(1)*(a - b* x(2));
    xdot(2) = -x(2)*(c - d*x(1));
endfunction
```

\(\frac{dx}{dt} = f(x, t)\)
Set initial conditions: \( x_0 = [5; 5] \)

\[
t = \text{linspace}(0, 500, 1000);
\]
\[
y = \text{lsode}("vp", x0, t);
\]

Plotting is done with:

\[
\text{plot}(t,y)
\]

Change ICs. What do you see?
Octave 1st ODE (3)

For convenience, functions can be placed in files.

Example: vp.m (the “m” extension is for compatibility w/ MATLAB)

```octave
# example of function file
function xdot = vp(x,t)
  xdot = zeros(2,1);
  a = 1.0;
  b = 0.5;
  c = 0.95;
  d = 0.25;
  xdot(1) = x(1)*(a - b* x(2));
  xdot(2) = -x(2)*(c - d*x(1));
endfunction
```

Usage: octave:41> vp

Note: octave will throw errors (b/c x is not specified) . Ignore them.
Planetary Orbits via octave

Solve for Earth's motion around the Sun using octave.

Produce a plot showing orbit!!
Summary

Gaussian elimination.
LU decomposition
octave intro
Finding eigenvalues and eigenvectors
octave calisthenics
Solving ODEs w/ octave

Don’t suffer in silence. Scream for help!!!