Lecture 4 Review

- Representation of single & double precision real numbers.
- Either representation has a finite precision.
- Code to determine machine precision for single precision numbers.
- Example of variable declaration (single precision real).
- Example of `for` loop.
- Simple example of a “here document” in shell scripting.

http://www.cplusplus.com/doc/tutorial/variables.html

http://www.cplusplus.com/doc/tutorial/control.html

Numerical Integration

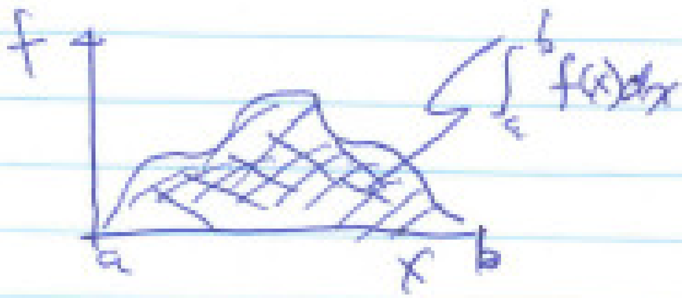
NUMERICAL INTEGRATION

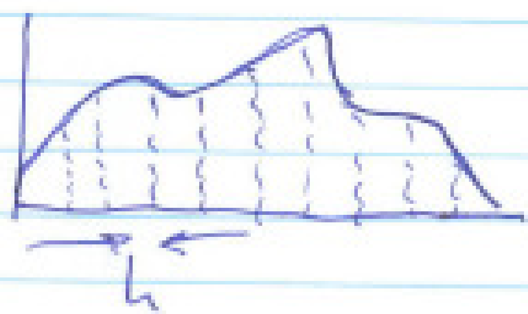
NEED TO PERFORM (NUMERICAL) INTEGRATION FROM TIME-TO-TIME.

$$\int_a^b f(x)\,dx = A$$

BASIC TECHNIQUE IS STRAIGHTFORWARD

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left[h\sum_{i=1}^{(b-a)/h} f(x_i)\right]$$





Numerical Integration (2)

MANY DIFFERENT INTEGRATION TECHNIQUES.

BASIC IDEA IS TO "SUM OVER BOXES"

MORE ABSTRACTLY,

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{N} f(x_i)\, w_i$$

"WEIGHTS"

ACTUALLY, NO REAL NEED TO HAVE BOXES ALL OF SAME WIDTH. WIDTH CAN VARY.

HOWEVER, LET'S START SIMPLY.

$$h = \frac{b-a}{N-1}$$ BOX WIDTHS ALL EQUAL

$$x_i = a + (i-1)h, \quad i = 1, N$$

WE CONSTRUCT TRAPEZOIDS.

Numerical Integration (3)

CONSIDER AREA OF <u>SINGLE</u> TRAPEZOID

$$\int_{x_i}^{x_i+h} f(x)\,dx \simeq \frac{h(f_i + f_{i+1})}{2} = \tfrac{1}{2}h f_i + \tfrac{1}{2}h f_{i+1}$$

IN TERMS OF OUR STD INTEGRATION FORMULA

$$\int_a^b f(x)\,dx \simeq \sum_1^N f(x_i)\,w_i \quad , \quad w_i \equiv 1/2$$

FOR THE FULL RANGE $[a, b]$,
ADD CONTRIBUTIONS FROM EACH SUB-INTERVAL

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}f_1 + \frac{h}{2}f_2 + \frac{h}{2}f_2 + \frac{h}{2}f_3 + \frac{h}{2}f_3 + \ldots \frac{h}{2}f_N$$

<u>NOTE</u>: EACH INTERNAL POINT COUNTS TWICE

HENCE, $w_i = \left\{ \frac{h}{2}, h, \ldots h, \frac{h}{2} \right\}$

MAIN ADVANTAGE: EASY TO UNDERSTAND
EASY TO CODE

$\boxed{\text{N MUST BE ODD}}$

Don't Forget

SLIGHTLY FANCIER TECHNIQUE: SIMPSON'S RULE

TRIES TO ACCOUNT FOR CURVINESS OF $f(x)$
FOR SMALL INTERVALS $f(x) \simeq \alpha x^2 + \beta x + \gamma$

STILL KEEP INTERVALS EQUALLY SPACED.

WILL EVALUATE INTEGRAL OVER ADJACENT INTERVALS (AS BEFORE).
N MUST BE ODD

AREA OF EACH INTERVAL IS INTEGRAL OF PARABOLA

$$\int_{x_i}^{x_i+h} (\alpha x^2 + \beta x + \gamma)\,dx = \frac{\alpha x^3}{3} + \frac{\beta x^2}{2} + \gamma x \Big|_{x_i}^{x_i+h}$$

NOW, CONSIDER INTERVAL $[-1,1]$

$$\int_{-1}^{1} (\alpha x^2 + \beta x + \gamma)\,dx = \frac{2\alpha}{3} + 2\gamma$$

Numerical Integration (5)

ALSO NOTICE $f(-1)=\alpha-\beta+\gamma \quad f(0)=\gamma \quad f(1)=\alpha+\beta+\gamma$

$$\Rightarrow \alpha = \frac{f(1)+f(-1)}{2} - f(0)$$

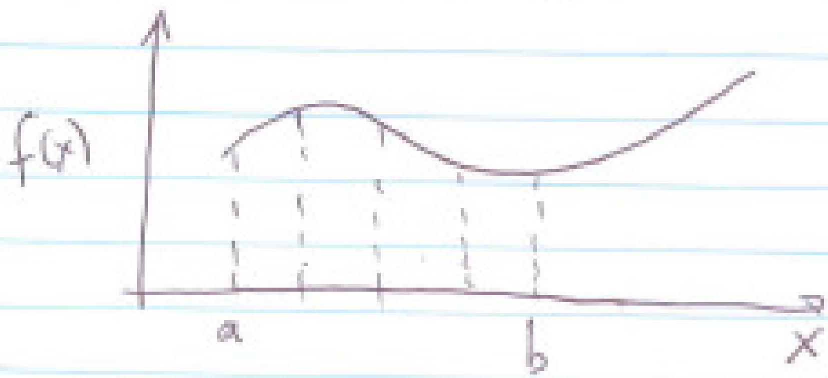
$$\beta = \frac{f(1)+f(-1)}{2} \qquad \gamma = f(0)$$

SO,

$$\int_{-1}^{1}(\alpha x^2+\beta x+\gamma)\,dx = \frac{f(-1)}{3} + \frac{4f(0)}{3} + \frac{f(1)}{3}$$

EXPAND TO CONSIDER 2 ADJACENT INTERVALS

$$\int_{x_i-h}^{x_i+h} f(x)\,dx = \int_{x_i}^{x_i+h} f(x)\,dx + \int_{x_i-h}^{x_i} f(x)\,dx \simeq \frac{h}{3} f_{i-1} \quad \frac{4h}{3} f_i + \frac{h}{3} f_{i+1}$$



NOTE: $N = \text{ODD}$

Don't Forget

Numerical Integration (6)

So, ...

$$\boxed{\int_a^b f(x)\,dx \cong \frac{h}{3}f_1 + \frac{4h}{3}f_2 + \frac{2h}{3}f_3 + \frac{4h}{3}f_4 + \frac{2h}{3}f_5 + \ldots \frac{4h}{3}f_{N-1} + \frac{h}{3}f_N}$$

IN TERMS OF OUR WEIGHT FORMULA

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} f(x_i)\, w_i$$

$$w_i = \underbrace{\left\{ \frac{h}{3}, \frac{4h}{3}, \frac{2h}{3}, \frac{4h}{3}, \frac{2h}{3}, \ldots \frac{4h}{3}, \frac{h}{3} \right\}}_{N \text{ WEIGHTS}}$$

N WEIGHTS, N = ODD

BTW, EASY TO SHOW $\sum_{i=1}^{N} w_i = (N-1)h$

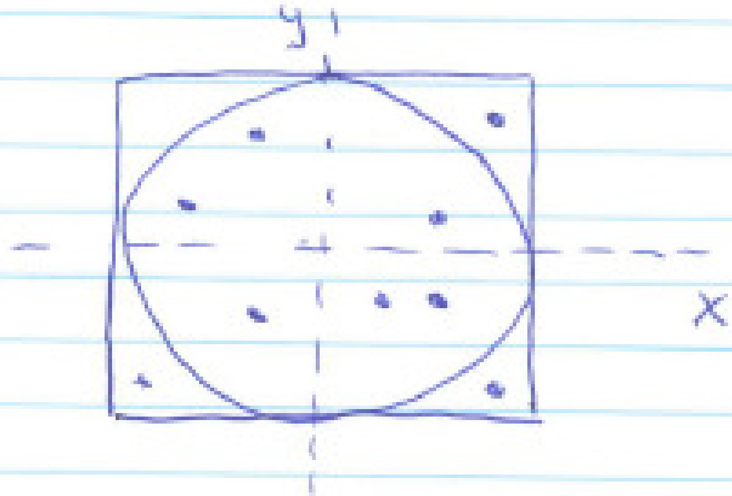
Simpson's Rule

Display simpson's rule source code (too large for 1 slide).

http://www.physics.smu.edu/devel/coan/3340/simpson.cc

MONTE CARLO INTEGRATION

THROW DARTS AT DART BOARD



PROBABILITY OF LANDING IN CIRCLE
PROPORTIONAL TO AREA OF CIRCLE

BUT AREA = INTEGRAL UNDER CURVE

CONSIDER UPPER QUADRANT $0 \le x \le 1$, $0 \le y \le 1$
AREA IN QUAD = $\int y \, dx$

MC INTEGRATION (2)

$$A = \int_0^1 \sqrt{1-x^2}\, dx$$

"THROW A DART" MEANS PICK RANDOM NMBR

RANDOM ≡ CAN'T BE PREDICTED.

PRACTICALLY, DONE BY FUNCTION CALL.

PICK RANDOM X $(0 \le x < 1)$

PICK RANDOM y $(0 \le y \le 1)$

IF $y \le \sqrt{1-x^2}$, THEN DART IN QUADRANT

REPEAT MANY TIMES, SAY, N_T

CALCULATE HOW MANY DARTS -CIRCLE N_u

$$A = \left(N_u / N_T\right) * \text{AREA OF SQUARE}$$

Random Number Generation

```
// computes some random numbers

#include <iostream>

using namespace std;

int main()
{
  srand((time(0)));  // "seed" the random number generator
  int r = random();

  // RAND_MAX is largest random number. it is an integer. built-in.

  cout << " random number: " << (double) random()/RAND_MAX <<
endl;
  cout << " random number: " << random() << endl;
  cout << " random number: " << random() << endl;

cout << "The value of RAND_MAX is " <<  RAND_MAX << endl;
  return 0;
}
```

Help with C++ Control Structure and Math

My head is exploding. 

I need something to read quietly, at my <u>own</u> pace.

http://www.cplusplus.com/doc/tutorial/control.html

http://www.cplusplus.com/reference/clibrary/cmath/

Link also available from PHYS 3340 links page

Summary

- Trapezoid and Simpson's rule for numerical integration.
- Monte Carlo ("dart throwing") technique for integration.
- Code for random number generation.
- C++ features: `if`, declaring functions, `random`, ...
- Just a peek at plotting: gnuplot.

Don't suffer in silence. Scream for help!!!