

2003 CTEQ Summer School

Drell-Yan Process Lecture #3

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(SMU)

For lecture 3, we will examine a NLO DY calculation. We will be using the notes from C.P. Yuan (beautifully typeset by Qing-Hong Cao), and supplement this with examples from FeynCalc and CompHEP

The sample files used will be posted after the school. In the meantime, refer to the attached lecture, and *Mathematica notebooks of FeynArts and FeynCalc sample calculations* which are on the web at cteq.org under the Miscellaneous heading.

A NLO Calculation of pQCD: Total Cross Section of $P\bar{P} \rightarrow W^+ + X$

C.-P. Yuan

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CTEQ Summer School, June 2002

Outline

1. Parton Model
⇒ Born Cross Section
2. Factorization Theorem
⇒ How to organize a NLO calculation of pQCD
3. Feynman rules and Feynman diagrams
⇒ "Cut diagram" notation
4. Immediate Problems (Singularities)
⇒ Dimensional Regularization
5. Virtual Corrections
6. Real Emission Contribution
7. Perturbative Parton Distribution Functions
8. Summary of NLO [$O(\alpha_s)$] Corrections

Appendices:

- A. γ -matrices in n dimensions
- B. Some integrals and "special functions"
- C. Angular integrals in n dimensions
- D. Two-particle phase space in n dimensions
- E. Explicit Calculations

(Typesetting: prepared by Qing-Hong Cao at MSU.)

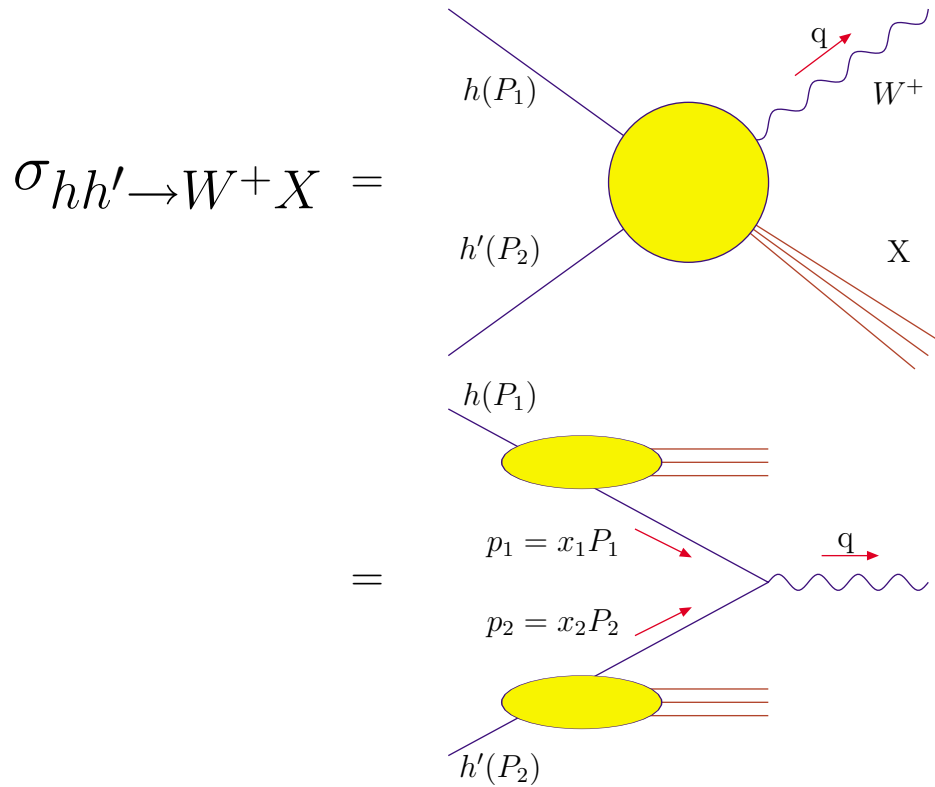
A few references can be found in

"Handbook of pQCD"

on **CTEQ** website

<http://www.phys.psu.edu/~cteq/>

Parton Model



$$\sigma_{hh' \rightarrow W^+ X} = \sum_{f, f' = q, \bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1) \hat{\sigma}_{ff'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

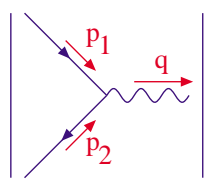
Partonic "Born"
Cross Section of $f\bar{f}' \rightarrow W^+$

The probability of finding a "parton" f with fraction x_1 of the hadron h momentum

Born Cross Section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3 2q_0} (2\pi)^4 \delta^4(p_1 + p_2 - q) \cdot \overline{|\mathcal{M}|^2}$$

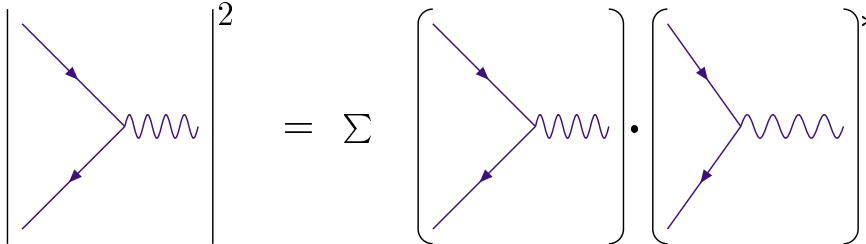
where

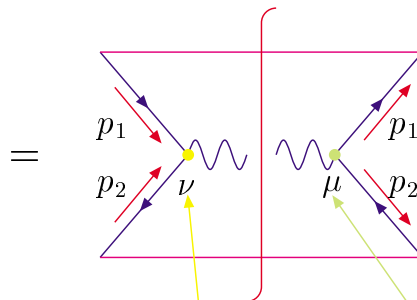
$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right)}_{\text{average color and spin}} \sum_{\text{spin color}} \left| \text{Diagram} \right|^2$$


average color and spin

$$\left[\text{Or, } -i\mathcal{M} = \bar{v}(p_2) \frac{ig_w}{\sqrt{2}} \gamma_\mu \frac{1}{2} (1 - \gamma_5) u(p_1) \right]$$

"Cut-diagram" notation

$$\Sigma \left| \text{Diagram} \right|^2 = \Sigma \left[\text{Diagram} \right] \cdot \left[\text{Diagram} \right]^*$$




$$\frac{ig_w}{\sqrt{2}} \gamma_\nu P_L$$

$$-\frac{ig_w}{\sqrt{2}} \gamma_\mu P_L$$

$$P_L \equiv \frac{1}{2}(1 - \gamma_5)$$

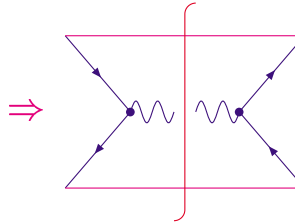
$$= \left(\frac{g_w}{\sqrt{2}}\right)^2 \text{Tr} [\not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma_\nu P_L] \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \cdot \text{Tr} I_{3 \times 3}$$

Doesn't contribute for $m_q = 0$,
due to Ward identity

Color

$$\begin{aligned}
& Tr [\not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L] (-1) \\
&= Tr [\not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L] (-1) & P_L P_L = P_L = \frac{1}{2} (1 - \gamma_5) \\
&= (-2) Tr [\not{p}_1 \not{p}_2 P_L] (-1) & \gamma_\mu \not{p}_2 \gamma^\mu = -2 \not{p}_2 \\
&= (-2) \cdot \frac{1}{2} \cdot 4 (p_1 \cdot p_2) (-1) & Tr (\not{p}_1 \not{p}_2) = 4 (p_1 \cdot p_2) \\
&= +2\hat{s} & Tr (\not{p}_1 \not{p}_2 \gamma_5) = 0
\end{aligned}$$

$$Tr [I_{3 \times 3}] = 3 \quad (\hat{s} \equiv (p_1 + p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0)$$

$$\Rightarrow \text{Diagram} = \left(\frac{g_w}{\sqrt{2}} \right)^2 \cdot (+2\hat{s}) (3) = 3 g_w^2 \hat{s}$$


$$\begin{aligned}
\int \frac{d^3 q}{2q_0} \delta^4 (p_1 + p_2 - q) &= \int d^4 q \delta^4 (p_1 + p_2 - q) \delta^+ (q^2 - M^2) \\
&= \delta (q^2 - M^2)
\end{aligned}$$

where M is the mass of W -boson.

Thus,

$$\begin{aligned}
\hat{\sigma}_{q\bar{q}'} &= \frac{1}{2\hat{s}} (2\pi) \cdot \delta (\hat{s} - M^2) \cdot \left(\frac{1}{3} \right) \left(\frac{1}{2} \cdot \frac{1}{2} \right) \cdot g_w^2 \hat{s} \\
&= \frac{\pi}{12} g_w^2 \delta (\hat{s} - M^2) \\
&= \frac{\pi}{12\hat{s}} g_w^2 \delta (1 - \hat{\tau})
\end{aligned}$$

$$\left(\begin{array}{l} \hat{\tau} = M^2/\hat{s}, \hat{s} = x_1 x_2 S \text{ for} \\ S = (P_1 + P_2)^2 \text{ and } P_1^2 = P_2^2 = 0 \end{array} \right)$$

Factorization Theorem

$$\sigma_{hh'} = \sum_{i,j} \int_0^1 dx_1 dx_2 \phi_{i/h}(x, Q^2) H_{ij} \left(\frac{Q^2}{x_1 x_2 S} \right) \phi_{j/h'}(x_2, Q^2)$$

Nonperturbative,
but universal,
hence, measurable

IRS, Calculable
in pQCD

Procedure:

- (1) Compute σ_{kl} in pQCD with k, l partons
(not h, h' hadron)

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \phi_{i/k}(x, Q^2) H_{ij} \left(\frac{Q^2}{x_1 x_2 S} \right) \phi_{j/l}(x_2, Q^2)$$

- (2) Compute $\phi_{i/k}, \phi_{j/l}$ in pQCD

- (3) Extract H_{ij} in pQCD

$$\begin{aligned} H_{ij} \text{ IRS} &\Rightarrow H_{ij} \text{ independent of } k, l \\ &\Rightarrow \text{same } H_{ij} \text{ with } (k \rightarrow h, l \rightarrow h') \end{aligned}$$

- (4) Use H_{ij} in the above equation with $\phi_{i/h}, \phi_{j/h'}$

Extracting H_{ij} in pQCD

- Expansions in α_s :

$$\sigma_{kl} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \sigma_{kl}^{(n)} \quad \alpha_s = \frac{g^2}{4\pi}$$

$$H_{ij} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n H_{ij}^{(n)}$$

$$\phi_{i/k}(x) = \delta_{ik} \delta(1-x) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \phi_{i/k}^{(n)}$$

\uparrow
 $\phi_{i/k}^{(0)}$ ($\alpha_s = 0 \Rightarrow$ Parton k “ stays itself ”)

- Consequences:

$$H_{ij}^{(0)} = \sigma_{ij}^{(0)} = \text{“Born”}$$

suppress “^” from now on

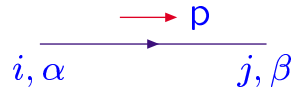
$$H_{ij}^{(1)} = \sigma_{ij}^{(1)} - \left[\sigma_{il}^{(0)} \phi_{l/j}^{(1)} + \phi_{k/i}^{(1)} \sigma_{kj}^{(0)} \right]$$

Computed from
 Feynman diagrams
 (process dependent)

Computed from
 the definition of
 perturbative parton
 distribution function
 (process independent,
 scheme dependent)

Feynman Rules

- Quark Propagator

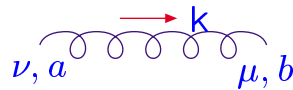


$$\frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}\delta_{ij}$$

(i,j=1,2,3)

Take $m=0$ in our calculation

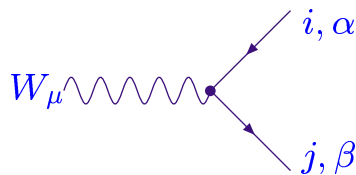
- Gluon Propagator



$$\frac{i(-g_{\mu\nu})}{k^2+i\epsilon}\delta_{ab}$$

(a,b=1,2,...,8)

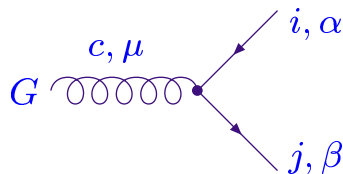
- Quark-W Vertex



$$i\frac{g_W}{\sqrt{2}}(\gamma_\mu)_{\beta\alpha}\frac{(1-\gamma_5)}{2}\delta_{ij}$$

$$g_w = \frac{e}{\sin\theta_w}, \text{ weak coupling}$$

- Quark-Gluon Vertex



$$-ig(t_c)_{ji}(\gamma_\mu)_{\beta\alpha}$$

t_c is the $SU(N)_{N\times N}$ generator

- Quark Color Generators

$$[t_a, t_b] = if_{abct}t_c$$

$$\sum_c t_c^2 = C_F I_{N\times N}$$

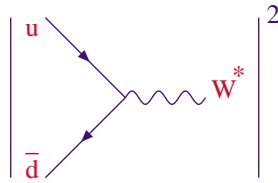
$$Tr(\sum_c t_c^2) = N C_F$$

$$C_F = \frac{N^2 - 1}{2N} = \frac{4}{3}, \quad (N = 3)$$

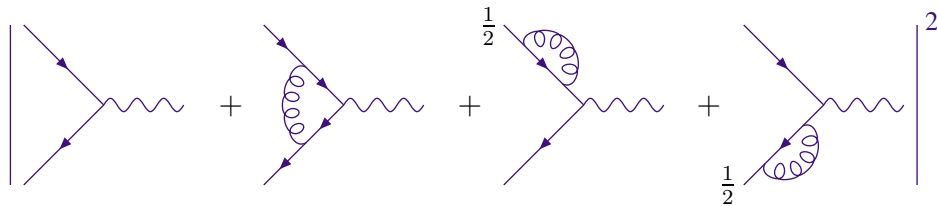
Feynman Diagrams

- Born level

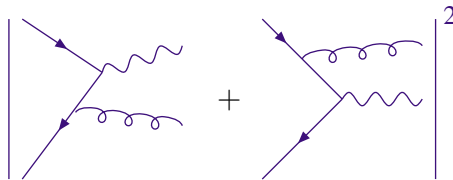
$$\alpha_s^{(0)} \quad (q\bar{q}')_{Born}$$



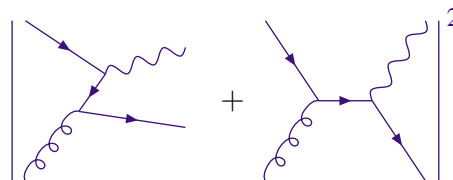
- NLO: $(\alpha_s^{(1)})$ virtual corrections $(q\bar{q}')_{virt}$



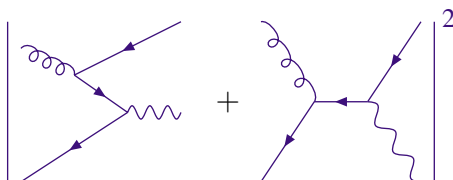
- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(q\bar{q}')_{real}$



- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(qG)_{real}$

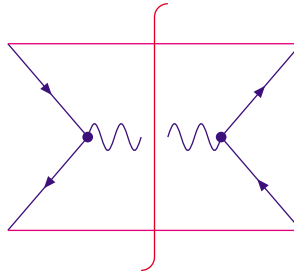


- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(G\bar{q}')_{real}$

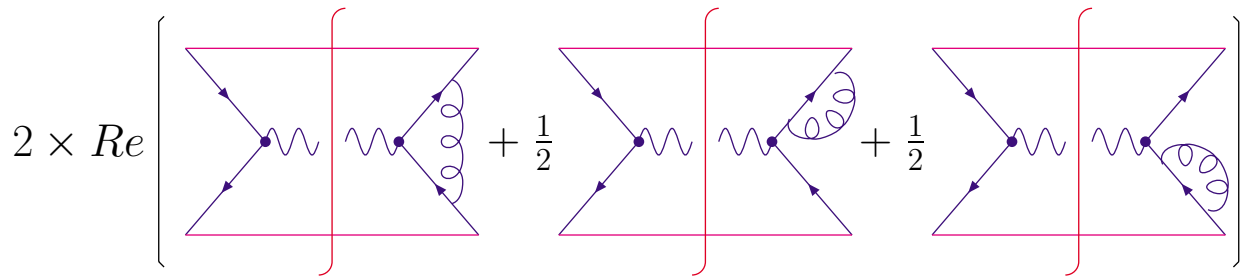


In "Cut-diagram" notation

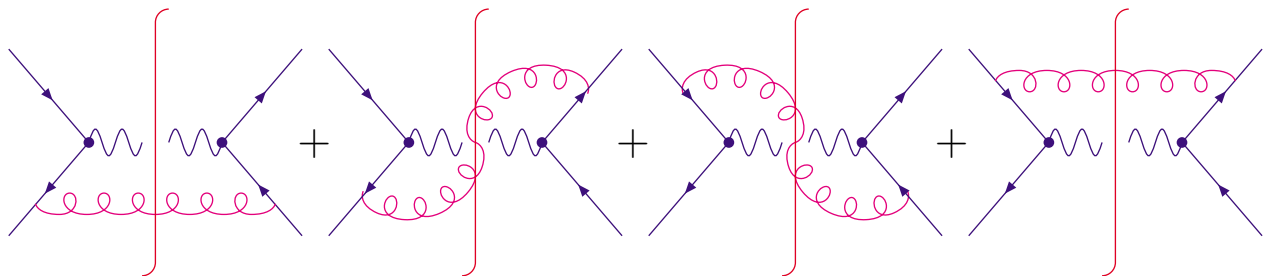
- $(q\bar{q}')_{Born}$



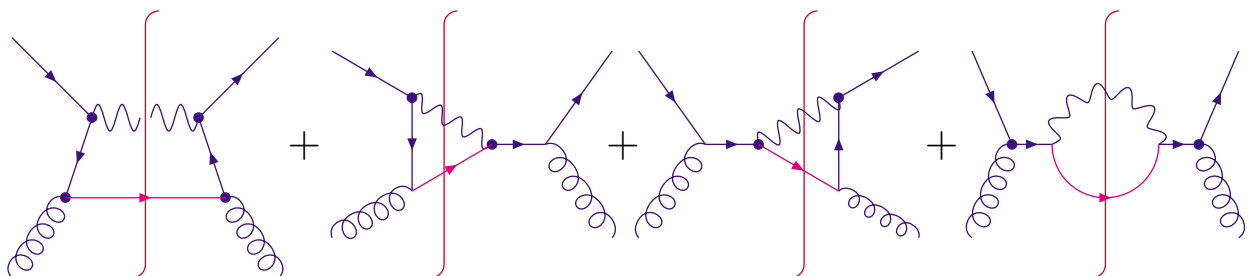
- $(q\bar{q}')_{virt}$



- $(q\bar{q}')_{real}$



- $(qG)_{real}$

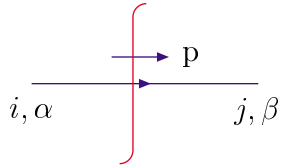


- $(G\bar{q}')_{real}$

Same as $(qG)_{real}$ after replacing q by \bar{q}' .

Feynman rules for cut-diagrams

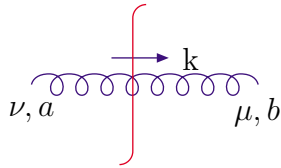
- quark line



$$(2\pi)\delta^+(p^2 - m^2)(\not{p} + m)_{\beta\alpha}\delta_{ij}$$

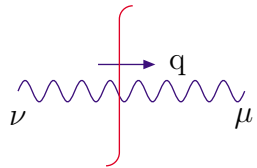
$\delta(p^2 - m^2)\theta(p_0)$

- gluon line



$$(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}$$

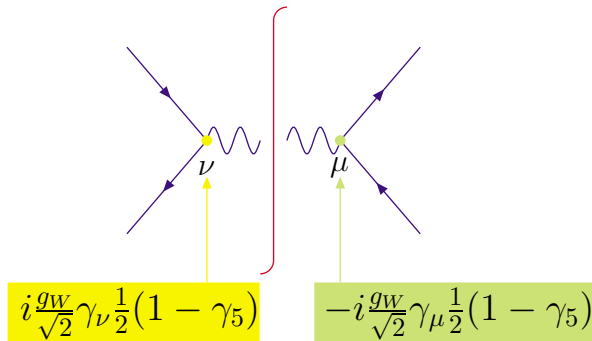
- W-boson line



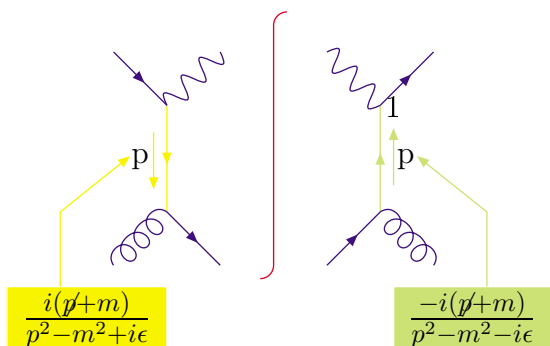
$$(2\pi)\delta^+(q^2 - M^2)(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2})$$

Doesn't contribute for $m_f = 0$ because of Ward identity

-



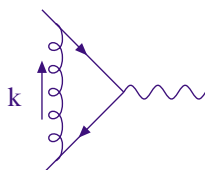
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Immediate problems (Singularities)

- Ultraviolet singularity

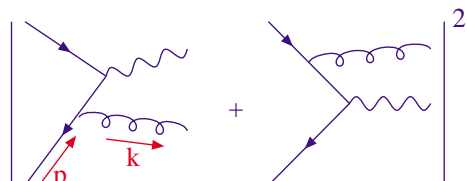
(UV)



$$\sim \int d^4k \frac{k \cdot k}{(k^2)(k^2)(k^2)} \rightarrow \infty$$

- Infrared singularities

(IR)



$$\rightarrow \infty$$

as $k^\mu \rightarrow 0$ (soft divergence)

or $k^\mu \parallel p^\mu$ (collinear divergence)

(Similar singularities also exist in virtual diagrams.)

- Solutions

Compute H_{ij} in pQCD in $n = 4 - 2\varepsilon$ dimensions
(dimensional regularization)

(1) $n \neq 4 \Rightarrow$ UV & IR divergences appear as $\frac{1}{\varepsilon}$ poles
in $\sigma_{ij}^{(1)}$ (Feynman diagram calculation)

(2) H_{ij} is IR safe \Rightarrow no $\frac{1}{\varepsilon}$ in H_{ij}

(H_{ij} is UV safe after "renormalization".)

Dimensional Regularization

(Revisit the Born Cross Section in n dimensions)

- $$\hat{\sigma}_{q\bar{q}'}^{(0)} = \frac{1}{2\hat{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1} 2q_0} (2\pi)^n \cdot \delta^n(p_1 + p_2 - q) \cdot \overline{|m|^2}$$

- $$\overline{|m|^2} = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\text{diagram} \right)$$

In n -dim, the polarization degree of freedom is (2) for a quark, and $(n-2)$ for a gluon.

- Using the Naive- γ^5 prescription:

$$\begin{aligned} & Tr [\not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L] (-1) \\ &= Tr [\not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L] (-1) && \gamma_\mu \not{p}_2 \gamma^\mu = -2(1-\epsilon) \not{p}_2 \\ &= (-2)(1-\epsilon) Tr [\not{p}_1 \not{p}_2 P_L] (-1) \\ &= (-2)(1-\epsilon) \cdot \frac{1}{2} \cdot 4(p_1 \cdot p_2) (-1) \\ &= 2(1-\epsilon) \hat{s} \end{aligned}$$

- In n dimensions

$$\hat{\sigma}_{q\bar{q}'}^{(0)} = \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\epsilon) \cdot \delta(1-\hat{\tau})$$

Strong Coupling g in n dimensions

- In n dimensions

$$\int d^n x \mathcal{L}$$
$$\longrightarrow \int d^n x \left\{ \bar{\psi} i \not{\partial} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \dots \right\}$$

The dimension in mass unit (μ)

$$[\psi] \sim \mu^{\frac{n-1}{2}}$$

$$[G] \sim \mu^{\frac{n-2}{2}}$$

$$[\bar{\psi} G \psi] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

Since $[g \bar{\psi} G \psi] \sim \mu^n$, so

$$[g] \sim \mu^{\frac{-n}{2} + 2} \quad n = 4 - 2\varepsilon$$
$$= \mu^\varepsilon$$

$\Rightarrow g$ has a dimension in mass when $\varepsilon \neq 0$

\Rightarrow Feynman rules should read $g \rightarrow g \mu^\varepsilon$

Calculations

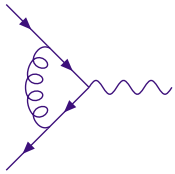
- Tools needed for a NLO calculation are collected in Appendices A-D
- The detailed calculation for each subprocess can be found in Appendices E
- In the following, I shall summarize the result for each subprocess

Virtual Corrections $(q\bar{q}')_{virt}$ (in Feynman Gauge)



$$= 0$$

$\frac{1}{\epsilon_{IR}}$ and $\frac{1}{\epsilon_{UV}}$ poles cancel when $\epsilon_{UV} = -\epsilon_{IR} \equiv \epsilon$



$$\frac{1}{\epsilon_{UV}}$$

cancel \Rightarrow Electroweak coupling is not renormalized by QCD interactions at one-loop order
(Ward identity, a renormalizable theory)

$$\frac{1}{\epsilon_{IR}}$$

poles remain

$\sigma_{virt}^{(1)}$ is free of ultraviolet singularity.

$$\sigma_{virt}^{(1)} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta(1 - \hat{\tau}) \left(\frac{4\pi\mu^2}{M^2} \right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \cdot \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3} \right\} \cdot (C_F)$$

$-\frac{2}{\epsilon^2}$: soft and collinear singularities

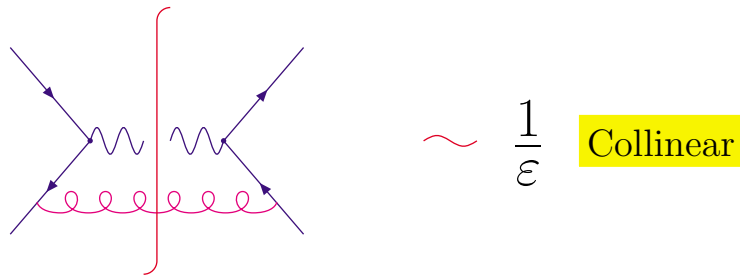
$-\frac{3}{\epsilon}$: soft or collinear singularities

C_F : color factor

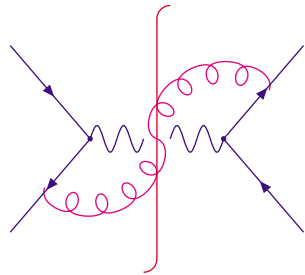
$$\sigma^{(0)} \equiv \frac{\pi}{12\hat{s}} g_w^2$$

Real Emission Contribution $(q\bar{q}')_{real}$

•



$$\sim \frac{1}{\epsilon} \quad \text{Collinear}$$



$$\sim \frac{1}{\epsilon^2} \quad \text{Soft and Collinear}$$

•

$$\sigma_{\text{real}}^{(1)}(q\bar{q}') = \sigma^{(0)} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \cdot C_F \cdot \left\{ \frac{2}{\epsilon^2} \delta(1-\hat{\tau}) - \frac{2}{\epsilon} \frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4(1+\hat{\tau}^2) \left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}} \right)_+ - 2 \frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} \right\}$$

Note: $[\dots]_+$ is a distribution,

$$\int_0^1 dz f(z) \left[\frac{1}{1-z} \right]_+ = \int_0^1 dz \frac{f(z) - f(1)}{1-z}, \text{ which is finite.}$$

• In the soft limit, $\hat{\tau} \rightarrow 1$ ($\hat{\tau} = \frac{M^2}{\hat{s}}$),

$$\sigma_{\text{real}}^{(1)}(q\bar{q}') \rightarrow \sigma^{(0)} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} \cdot C_F \cdot \left\{ \frac{2}{\epsilon^2} \delta(1-\hat{\tau}) - \frac{4}{\epsilon} \frac{1}{(1-\hat{\tau})_+} + 8 \left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}} \right)_+ \right\}$$

$(q\bar{q}')_{virt} + (q\bar{q}')_{real}$ at NLO

$$\begin{aligned}
 \sigma_{q\bar{q}'}^{(1)} &= \sigma_{virt}^{(1)}(q\bar{q}') + \sigma_{real}^{(1)}(q\bar{q}') \\
 &= \sigma^{(0)} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2} \right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \cdot C_F \\
 &\quad \cdot \left\{ \frac{-2}{\varepsilon} \left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \right)_+ - 2 \frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 4(1+\hat{\tau}^2) \left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}} \right)_+ \right. \\
 &\quad \left. + \left(\frac{2\pi^2}{3} - 8 \right) \delta(1-\hat{\tau}) \right\}
 \end{aligned}$$

Where we have used

$$\frac{-2}{\varepsilon} \left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2} \delta(1-\hat{\tau}) \right] = \frac{-2}{\varepsilon} \left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \right)_+$$

All the soft singularities $(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon})$ cancel in $\sigma_{q\bar{q}'}^{(1)}$

\Rightarrow *KLN* theorem

(Kinoshita-Lee-Navenberg)

$$\sigma_{q\bar{q}'}^{(1)} \sim \frac{1}{\varepsilon} (\text{term}) + \text{finite (terms)}$$

↑

Collinear Singularity

Factorization Theorem

- Perturbative PDF

$$\phi_{i/k}^{(0)} = \delta_{ik} \delta(1-x)$$

$\frac{\alpha_s}{\pi} \phi_{i/k}^{(1)}$ can be calculated from the definition of PDF.

(Process independent, but factorization scheme dependent)

- (1)

$$\sigma_{kl}^{(0)} = \begin{array}{c} k \\ \swarrow \\ \textcircled{\phi_{i/k}^{(0)}} \\ \searrow \\ i \\ \swarrow \\ \textcircled{H_{ij}^{(0)}} \\ \searrow \\ j \\ \swarrow \\ \textcircled{\phi_{j/l}^{(0)}} \\ \searrow \\ l \end{array} \Rightarrow H_{kl}^{(0)} = \sigma_{kl}^{(0)}$$

- (2)

$$\sigma_{kl}^{(1)} = \begin{array}{c} k \\ \swarrow \\ \textcircled{\phi_{i/k}^{(1)}} \\ \searrow \\ i \\ \swarrow \\ \textcircled{H_{ij}^{(0)}} \\ \searrow \\ j \\ \swarrow \\ \textcircled{\phi_{j/l}^{(0)}} \\ \searrow \\ l \end{array} + \begin{array}{c} k \\ \swarrow \\ \textcircled{\phi_{i/k}^{(0)}} \\ \searrow \\ i \\ \swarrow \\ \textcircled{H_{ij}^{(0)}} \\ \searrow \\ j \\ \swarrow \\ \textcircled{\phi_{j/l}^{(1)}} \\ \searrow \\ l \end{array} + \begin{array}{c} k \\ \swarrow \\ \textcircled{\phi_{i/k}^{(0)}} \\ \searrow \\ i \\ \swarrow \\ \textcircled{H_{ij}^{(1)}} \\ \searrow \\ j \\ \swarrow \\ \textcircled{\phi_{j/l}^{(0)}} \\ \searrow \\ l \end{array}$$

Factorization scheme dependent

$$\Rightarrow H_{kl}^{(1)} = \sigma_{kl}^{(1)} - \left[\phi_{i/k}^{(1)} H_{il}^{(0)} + H_{kj}^{(0)} \phi_{j/l}^{(1)} \right]$$

Finite

Divergent

Perturbative PDF

- In \overline{MS} -scheme (modified minimal subtraction)

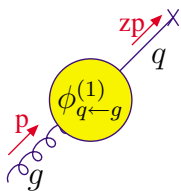
$$\phi_{q/q}^{(1)}(z) = \phi_{\bar{q}/\bar{q}}^{(1)}(z) = \frac{-1}{\varepsilon} \frac{1}{2} \left(4\pi e^{-\gamma_E}\right)^\varepsilon P_{q\leftarrow q}^{(1)}(z)$$

$$\phi_{q/g}^{(1)}(z) = \phi_{\bar{q}/g}^{(1)}(z) = \frac{-1}{\varepsilon} \frac{1}{2} \left(4\pi e^{-\gamma_E}\right)^\varepsilon P_{q\leftarrow g}^{(1)}(z)$$

where the splitting kernel for  is

$$P_{q\leftarrow q}^{(1)}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+ +$$

$$= C_F \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right)$$

and for  :

$$P_{q\leftarrow g}^{(1)}(z) = \frac{1}{2} \left(z^2 + (1-z)^2 \right)$$

(Note: The Pole part in the \overline{MS} scheme is $\frac{1}{\varepsilon} = \frac{1}{\varepsilon} (4\pi e^{-\gamma_E})^\varepsilon = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E$
 In the MS scheme, the pole part is just $\frac{1}{\varepsilon}$)

Find $H_{q\bar{q}'}^{(1)}$ (in the \overline{MS} scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{q\bar{q}'}^{(1)} = \sigma^{(0)} \left\{ P_{q\leftarrow q}^{(1)}(\hat{\tau}) \left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2(1+\tau^2) \left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right) \delta(1-\hat{\tau}) \right] \right\}$$



$$\begin{aligned} H_{q\bar{q}'}^{(1)}(\hat{\tau}) &= \sigma_{q\bar{q}'}^{(1)} - [2\phi_{q\leftarrow q}^{(1)}\sigma_{q\bar{q}'}^{(0)}] \\ &= \sigma^{(0)} \cdot \left\{ P_{q\leftarrow q}^{(1)}(\hat{\tau}) \ln\left(\frac{M^2}{\mu^2}\right) + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2(1+\tau^2) \left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right) \delta(1-\hat{\tau}) \right] \right\} \end{aligned}$$

where

$$\begin{aligned} \hat{\tau} &= \frac{M^2}{\hat{s}} = \frac{M^2}{x_1 x_2 S}, \\ \sigma^{(0)} &= \frac{\pi}{12\hat{s}} g_w^2 = \frac{\pi g_w^2}{12S} \frac{1}{x_1 x_2} \end{aligned}$$

- pQCD prediction

$$\begin{aligned} \sigma_{hh'} &= \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}(x_1, \mu^2) \left[\sigma^{(0)} \delta(1-\hat{\tau}) \right] \phi_{\bar{f}/h'}(x_2, \mu^2) \\ &+ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}(x_1, \mu^2) \left[\frac{\alpha_s(\mu^2)}{\pi} H_{f\bar{f}}^{(1)}(\hat{\tau}) \right] \phi_{\bar{f}/h'}(x_2, \mu^2) \\ &+ \left\{ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}(x_1, \mu^2) \left[\frac{\alpha_s(\mu^2)}{\pi} H_{fG}^{(1)}(\hat{\tau}) \right] \phi_{G/h'}(x_2, \mu^2) + (x_1 \leftrightarrow x_2) \right\} \end{aligned}$$

Find $H_{qG}^{(1)}$ (in the \overline{MS} scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{qG}^{(1)} = \sigma^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2P_{q \leftarrow g}^{(1)}(\hat{\tau}) \left[\frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] + \frac{3}{2} + \hat{\tau} - \frac{3}{2}\hat{\tau}^2 \right\}$$

•

$$\begin{aligned} H_{qG}^{(1)}(\hat{\tau}) &= \sigma_{qG}^{(1)} - [\sigma_{q\bar{q}'}^{(0)} \phi_{\bar{q}' \leftarrow G}^{(1)}] \\ &= \frac{\sigma^{(0)}}{2} \cdot \left\{ P_{q \leftarrow g}^{(1)}(\hat{\tau}) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \ln \left(\frac{(1-\hat{\tau})^2}{\hat{\tau}} \right) \right] + \frac{3}{4} + \frac{\hat{\tau}}{2} - \frac{3}{4}\hat{\tau}^2 \right\} \end{aligned}$$

- Similarly,

$$\begin{aligned} H_{G\bar{q}'}^{(1)} &= \sigma_{G\bar{q}'}^{(1)} - [\phi_{q \leftarrow G}^{(1)} \sigma_{q\bar{q}'}^{(0)}] \\ &= H_{qG}^{(1)} \end{aligned}$$

(Note: If we choose the renormalization scale $\mu^2 = M^2$,
then $\ln \left(\frac{M^2}{\mu^2} \right) = 0$)

Summary

- $\phi_{f/h}(x, \mu^2)$ depends on scheme (\overline{MS} , DIS, ...)
 $\Rightarrow H_{ij}$ **scheme dependent**

- Evolution equations allow us to predict
 q^2 -**dependent of** $\phi(x, q^2)$

- Essentially identical procedure for
 $hh' \rightarrow jets$, inclusive $Q\bar{Q}, \dots$

But, when the Born level process involves strong interaction (eg. $q\bar{q} \rightarrow t\bar{t}$), it is also necessary to renormalize the strong coupling α_s , etc, to eliminate ultraviolet singularities

Appendix A

γ -matrices in n dimensions

- Dirac algebra

$$\begin{aligned}\{\gamma^\mu, \gamma^\nu\}_+ &\equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \\ \mu, \nu &= 1, 2, \dots, n \quad g^{\mu\nu} = \text{diag}(1, -1, \dots, -1) \\ g^{\mu\nu} g_{\mu\nu} &= n \\ \{\gamma^\mu, \gamma^5\}_+ &= 0 \quad (\text{Naive-}\gamma^5\text{prescription})\end{aligned}$$

This works in calculating the inclusive rate of W -boson , but fails in the differential distributions of the leptons from the W -boson decay.

- Matrix identities

$$n = 4 - 2\varepsilon$$

$$\begin{aligned}\gamma_\mu \not{a} \gamma^\mu &= -2(1 - \varepsilon) \not{a} \\ \gamma_\mu \not{a} \not{b} \gamma^\mu &= 4a \cdot b - 2\varepsilon \not{a} \not{b} \\ \gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= -2 \not{c} \not{b} \not{a} + 2\varepsilon \not{a} \not{b} \not{c}\end{aligned}$$

- Traces

$$\begin{aligned}\text{Tr} [\not{a} \not{b}] &= 4(a \cdot b) \\ \text{Tr} [\not{a} \not{b} \not{c} \not{d}] &= 4\{(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\} \\ \text{Tr} [\gamma_5 \not{a} \not{b}] &= 0\end{aligned}$$

Appendix B

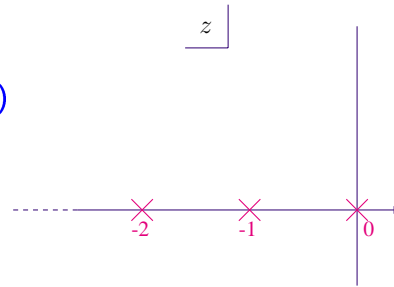
Some integrals and "special functions"

- The "Gamma function"

$$\Gamma(z) = \int_0^{\infty} dx x^{z-1} e^{-x} \quad (\operatorname{Re}(z) > 0)$$

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} \quad (\text{for all } z)$$

⇒ Poles in $\Gamma(z)$



$$\Gamma(n) = (n-1)! \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2} \left(\gamma_E^2 - \frac{\pi^2}{6} \right) + \dots$$

($\gamma_E = 0.5772\dots$, Euler constant)

$$\Gamma(1-\varepsilon) = -\varepsilon\Gamma(\varepsilon) = 1 + \varepsilon\gamma_E + \frac{1}{2}\varepsilon^2 \left(\frac{\pi^2}{6} + \gamma_E^2 \right) + O(\varepsilon^3)$$

$$\Gamma(1-\varepsilon)\Gamma(1+\varepsilon) = 1 + \varepsilon^2 \frac{\pi^2}{6} + O(\varepsilon^4)$$

$$z^\varepsilon = e^{\ln z^\varepsilon} = e^{\varepsilon \ln z} = 1 + \varepsilon \ln z + \dots$$

- The "Beta function"

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\begin{aligned} B(\alpha, \beta) &= \int_0^1 dy y^{\alpha-1} (1-y)^{\beta-1} = \int_0^{\infty} dy y^{\alpha-1} (1+y)^{-\alpha-\beta} \\ &= 2 \int_0^{\frac{\pi}{2}} d\theta (\sin \theta)^{2\alpha-1} (\cos \theta)^{2\beta-1} \end{aligned}$$

- Feynman trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax + b(1-x)]^2}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}}$$

- n-dimension integrals

$$\int d^n l \frac{l_\mu}{(l^2 - M^2)^\alpha} = 0$$

$$\int d^n l \frac{l_\mu l_\nu}{(l^2 - M^2)^\alpha} = \int d^n l \frac{\left(\frac{l^2 g_{\mu\nu}}{n}\right)}{(l^2 - M^2)^\alpha}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 - M^2)^\alpha} = \frac{(-1)^\alpha}{(4\pi)^{n/2}} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)} \left(\frac{1}{M^2}\right)^{\alpha - \frac{n}{2}}$$

$$\int d^n l \frac{l^2}{(l^2 - M^2)^\alpha} = \int d^n l \frac{(l^2 - M^2) + M^2}{(l^2 - M^2)^\alpha}$$

-

$$\text{Re} [(-1)^\epsilon] = 1 - \epsilon^2 \frac{\pi^2}{2} + O(\epsilon^4)$$

- "plus distribution" — to isolate $\frac{1}{\varepsilon}$ poles

Consider $\frac{1}{(1-z)^{1+2\varepsilon}}$

$$= \frac{1}{(1-z)^{1+2\varepsilon}} - \left[\delta(1-z) \int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} + \frac{1}{2\varepsilon} \delta(1-z) \right]$$

cancel

because $\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$ for $\varepsilon \rightarrow 0^-$

$$\equiv \left[\frac{1}{(1-z)^{1+2\varepsilon}} \right]_+ - \frac{1}{2\varepsilon} \delta(1-z)$$

$$= \frac{1}{(1-z)_+} - 2\varepsilon \left[\frac{\ln(1-z)}{1-z} \right]_+ + O(\varepsilon^2) - \frac{1}{2\varepsilon} \delta(1-z)$$

because $\frac{1}{(1-z)^{2\varepsilon}} = (1-z)^{-2\varepsilon}$
 $= 1 - 2\varepsilon \ln(1-z) + \dots$

- $[\dots]_+$ is a distribution

$$\int_0^1 dz f(z) \left[\frac{1}{1-z} \right]_+$$

$$\equiv \int_0^1 dz \frac{f(z)}{1-z} - \int_0^1 dz f(z) \delta(1-z) \int_0^1 \frac{dz'}{(1-z')}$$

$$= \int_0^1 dz \frac{f(z) - f(1)}{1-z}, \text{ which is finite.}$$

Appendix C

Angular integrals in n dimensions

- In n dimensions

$$\int d^n x = \int r^{n-1} d\Omega_{n-1}$$



$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1} \theta_{n-1} \int_0^\pi d\theta_{n-2} \sin^{n-2} \theta_{n-2} \cdots \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi$$

$$\Rightarrow \int d\Omega_1 = \int_0^{2\pi} d\phi \quad \longrightarrow \Omega_1 = 2\pi$$

$$\int d\Omega_2 = \int_0^\pi d\theta_1 \sin \theta_1 \int d\Omega_1 \quad \longrightarrow \Omega_2 = 4\pi$$

⋮

$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1} \theta_{n-1} \int d\Omega_{n-1}$$

$$\Rightarrow \Omega_n = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}$$

from repeated use of $B(\alpha, \beta)$

$$= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

because $\Gamma(n) = \frac{2^{n-1} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

Appendix D

Two-particle phase space in n dimensions

$$\int_{PS_2(p)} dk dq = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1} 2k_0} \frac{d^{n-1}\vec{q}}{(2\pi)^{n-1} 2q_0} \cdot (2\pi)^n \delta^n(p - q - k)$$

with $k^\mu = (k_0, \vec{k})$, etc.

Use $\frac{d^{n-1}\vec{q}}{2q_0} = \int d^n q \delta^+(q^2 - Q^2)$, we get

$$\begin{aligned} \int_{PS_2(p)} dk dq &= \frac{1}{(2\pi)^{n-2}} \int \frac{d^{n-1}\vec{k}}{2k_0} \delta^+((p-k)^2 - Q^2) \\ &= \frac{1}{(2\pi)^{n-2}} \int \frac{dk k^{n-3}}{2} \int d\Omega_{n-2} \delta(\hat{s} - 2k\sqrt{\hat{s}} - Q^2) \\ &\quad \left(p^2 \equiv \hat{s}, k^2 = 0, k = |\vec{k}| \right) \end{aligned}$$

Use $n = 4 - 2\varepsilon$, then in the c.m. frame $(p^\mu = (\sqrt{\hat{s}}, \vec{0}))$,

$$\int_{PS_2(p)} dk dq = \frac{\Omega_{n-3}}{(2\pi)^{2(1-\varepsilon)}} \int \frac{dk k^{1-2\varepsilon}}{4\sqrt{\hat{s}}} \int_0^\pi d\theta (\sin \theta)^{1-2\varepsilon} \cdot \delta\left(k - \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}}\right)$$

Use new variables:

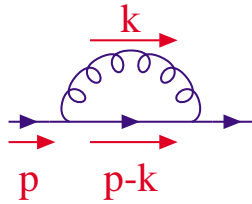
$$z = \frac{Q^2}{\hat{s}}, y = \frac{1}{2}(1 + \cos \theta) \Rightarrow k = \frac{\sqrt{\hat{s}}}{2}(1 - z),$$

$$\int_{PS_2(p)} dk dq = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\right)^\varepsilon \frac{z^\varepsilon (1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy [y(1-y)]^{-\varepsilon}$$

Appendix E

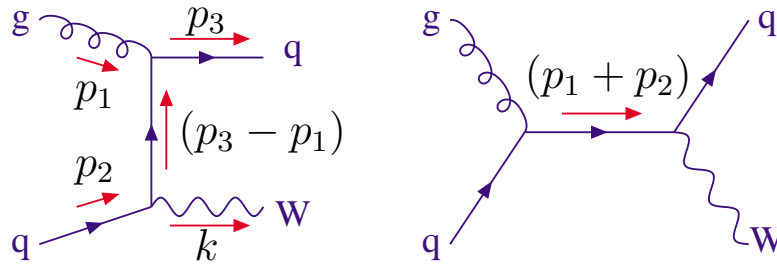
Explicit Calculations

Consider



$$\begin{aligned}
 & \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu (\not{p} - \not{k}) \gamma^\mu}{(k^2 + i\epsilon) ((p - k)^2 + i\epsilon)} \\
 \rightarrow & \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{(2 - n) (\not{p} - \not{k})}{[k^2 - 2k \cdot xp]^2} \quad (l \equiv k - xp) \\
 = & \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{(2 - n) [(1 - x) \not{p} - \not{l}]}{[l^2 + i\epsilon]^2} \\
 = & \left[\left(1 - \frac{n}{2}\right) \not{p} \right] \cdot \underbrace{\int \frac{d^n l}{(2\pi)^n} \frac{1}{[l^2 + i\epsilon]^2}} \\
 & \downarrow \\
 & = 0 \quad \left(\begin{array}{l} \text{Because there is} \\ \text{no mass scale} \end{array} \right) \\
 & \uparrow \\
 & \left(\begin{array}{l} \text{Due to cancellation} \\ \text{of } \frac{1}{\epsilon_{UV}} \text{ and } \frac{1}{\epsilon_{IR}} \\ \text{Trick: } A = A - B + B \end{array} \right) \\
 = & \int \frac{d^n l}{(2\pi)^n} \left\{ \underbrace{\left[\frac{1}{(l^2)^2} - \frac{1}{(l^2 - \Lambda^2)^2} \right]}_{\text{IR div.}} + \underbrace{\left[\frac{1}{(l^2 - \Lambda^2)^2} \right]}_{\text{UV div.}} \right\} \\
 = & \frac{i}{16\pi^2} \left(\frac{1}{\epsilon_{IR}} \right) + \frac{i}{16\pi} \left(\frac{1}{\epsilon_{UV}} \right), \quad \left(\begin{array}{l} n - 4 = 2\epsilon_{IR} \\ 4 - n = 2\epsilon_{UV} \end{array} \right)
 \end{aligned}$$

- consider the real emission process



Define the Mandelstam variables

$$\begin{aligned}\hat{s} &= (p_1 + p_2)^2 = 2p_1 \cdot p_2 \\ \hat{t} &= (p_1 - p_3)^2 = -2p_1 \cdot p_3 \\ \hat{u} &= (p_2 - p_3)^2 = -2p_2 \cdot p_3\end{aligned}$$

After averaging over colors and spins

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \underbrace{\left(\frac{1}{2(1-\varepsilon)} \frac{1}{2} \right)}_{\text{Spin}} \cdot \underbrace{\left(\frac{1}{3} \cdot \frac{1}{8} \right) \cdot \text{Tr}(t^a t^a)}_{\text{Color}} \cdot (g\mu^\varepsilon)^2 \\ &\quad \cdot g_w^2 \cdot 2(1-\varepsilon) \\ &\quad \cdot \left[(1-\varepsilon) \left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}} \right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon \right]\end{aligned}$$

Note: The d.o.f. of gluon polarization is $2(1-\varepsilon)$, and that of quark polarization is 2.

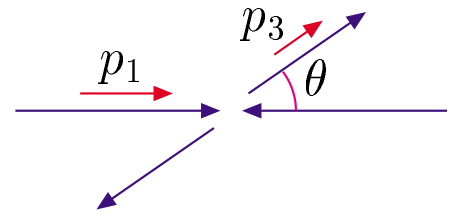
- In the **parton c.m.** frame, the constituent cross section

$$\begin{aligned}\hat{\sigma} &= \frac{1}{2\hat{s}} |\overline{\mathcal{M}}|^2 \cdot (PS_2) \\ &= \frac{1}{2\hat{s}} \cdot \left\{ \frac{1}{4} \cdot \frac{1}{6} \cdot 2g_s^2 \mu^{2\varepsilon} g_w^2 (1 - \varepsilon) \cdot \right. \\ &\quad \left. \left[(1 - \varepsilon) \left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}} \right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon \right] \right\} \\ &\quad \cdot \left\{ \frac{1}{8\pi} \left(\frac{4\pi}{M^2} \right)^\varepsilon \frac{1}{\Gamma(1 - \varepsilon)} \hat{\tau}^\varepsilon (1 - \hat{\tau})^{1-2\varepsilon} \int_0^1 dy [y(1 - y)]^{-\varepsilon} \right\}\end{aligned}$$

where $y \equiv \frac{1}{2}(1 + \cos\theta)$

Using $\hat{t} = -\hat{s} \left(1 - \frac{M^2}{\hat{s}} \right) (1 - y)$

$$\hat{u} = -\hat{s} \left(1 - \frac{M^2}{\hat{s}} \right) y$$



and

$$\int_0^1 dy y^\alpha (1 - y)^\beta = \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta)}{\Gamma(2 + \alpha + \beta)},$$

we get

$$\begin{aligned}\hat{\sigma}_{qG} &= \sigma^{(0)} \frac{\alpha_s}{4\pi} \cdot \left\{ 2P_{q \leftarrow g}^{(1)}(\hat{\tau}) \left[\frac{-1}{\varepsilon} \frac{\Gamma(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} + \ln \frac{M^2(1 - \hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] \right. \\ &\quad \left. + \frac{3}{2} + \hat{\tau} - \frac{3}{2}\hat{\tau}^2 \right\},\end{aligned}$$

with

$$\begin{aligned}P_{q \leftarrow g}^{(1)}(\hat{\tau}) &= \frac{1}{2} [\hat{\tau}^2 + (1 - \hat{\tau})^2] \\ \sigma^{(0)} &\equiv \frac{\pi}{12} g_w^2 \frac{1}{\hat{s}}\end{aligned}$$