A NLO Calculation of pQCD: Total Cross Section of $P\bar{P} \rightarrow W^+ + X$

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Outline

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E. Explicit Calculations

(TypeSETTING: prepared by Qing-Hong Cao at MSU.)

A few references can be found in

”Handbook of pQCD”

on CTEQ website

http://www.phys.psu.edu/~cteq/
\[ \sigma_{hh'\rightarrow W^+X} = \sum_{f,f'=q,\bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1) \tilde{\sigma}_{ff'} \phi_{f'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\} \]

The probability of finding a "parton" \( f \) with fraction \( x_1 \) of the hadron \( h \) momentum.
Born Cross Section

\[ \hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3} \frac{q_0}{2q_0} (2\pi)^4 \delta^4 (p_1 + p_2 - q) \cdot |\mathcal{M}|^2 \]

where

\[ |\mathcal{M}|^2 = \left( \frac{1}{3} \cdot \frac{1}{3} \right) \left( \frac{1}{2} \cdot \frac{1}{2} \right) \sum_{\text{spin}} \sum_{\text{color}} \]

average color and spin

\[ \text{Or,} \quad -i\mathcal{M} = \bar{v}(p_2) \frac{ig_w}{\sqrt{2}} \gamma_\mu \frac{1}{2} (1 - \gamma_5) u(p_1) \]

"Cut-diagram" notation

\[ \sum \left( \begin{array}{c} p_1 \\ q \\ p_2 \end{array} \right)^2 = \sum \left( \begin{array}{c} p_1 \\ q \\ p_2 \end{array} \right) \cdot \left( \begin{array}{c} p_1 \\ q \\ p_2 \end{array} \right) \]

\[ = \left( \frac{g_w}{\sqrt{2}} \right)^2 Tr \left[ \gamma_\mu P_L \gamma_\nu P_L \right] \cdot \left( -g^{\mu\nu} + \frac{q_0^2}{M^2} \right) \cdot Tr I_{3\times3} \]

 Doesn't contribute for \( m_q = 0 \), due to Ward identity
\[ \text{Tr} \left[ \not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L \right] (-1) \]
\[ = \text{Tr} \left[ \not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L \right] (-1) \]
\[ = (-2) \text{Tr} \left[ \not{p}_1 \not{p}_2 P_L \right] (-1) \]
\[ = (-2) \cdot \frac{1}{2} \cdot 4 (p_1 \cdot p_2) (-1) \]
\[ = +2 \hat{s} \]

\[ \text{Tr} \left[ I_{3 \times 3} \right] = 3 \quad \left( \hat{s} \equiv (p_1 + p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0 \right) \]

\[ \Rightarrow \quad \int \frac{d^3 q}{2q_0} \delta^4 (p_1 + p_2 - q) = \int d^4 q \delta^4 (p_1 + p_2 - q) \delta^+ \left( q^2 - M^2 \right) = \delta \left( q^2 - M^2 \right) \]

where \( M \) is the mass of \( W \)-boson.

Thus,
\[ \hat{\sigma}_{q \bar{q}} = \frac{1}{2\hat{s}} (2\pi) \cdot \delta \left( \hat{s} - M^2 \right) \cdot \left( \frac{1}{3} \right) \left( \frac{1}{2} \cdot \frac{1}{2} \right) \cdot g_w^2 \hat{s} \]
\[ = \frac{\pi}{12} g_w^2 \delta \left( \hat{s} - M^2 \right) \]
\[ = \frac{\pi}{12\hat{s}} g_w^2 \delta (1 - \hat{\tau}) \]

\[ \left( \hat{\tau} = M^2 / \hat{s} , \hat{s} = x_1 x_2 S \text{ for } S = (P_1 + P_2)^2 \text{ and } P_1^2 = P_2^2 = 0 \right) \]
**Factorization Theorem**

\[ \sigma_{hh'} = \sum_{i,j} \int_0^1 dx_1 dx_2 \phi_{i/h}(x, Q^2) H_{ij} \left( \frac{Q^2}{x_1 x_2 S} \right) \phi_{j/h'}(x_2, Q^2) \]

Nonperturbative, but universal, hence, measurable

IRS, Calculable in pQCD

**Procedure:**

1. **Compute** \( \sigma_{kl} \) in pQCD with \( k, l \) partons (not \( h, h' \) hadron)

   \[ \sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \phi_{i/k}(x_1, Q^2) H_{ij} \left( \frac{Q^2}{x_1 x_2 S} \right) \phi_{j/l}(x_2, Q^2) \]

2. **Compute** \( \phi_{i/k}, \phi_{j/l} \) in pQCD

3. **Extract** \( H_{ij} \) in pQCD

   \[ H_{ij} \text{ IRS } \Rightarrow H_{ij} \text{ independent of } k, l \]
   \[ \Rightarrow \text{ same } H_{ij} \text{ with } (k \to h, l \to h') \]

4. **Use** \( H_{ij} \) in the above equation with \( \phi_{i/h}, \phi_{j/h'} \)
Extracting $H_{ij}$ in pQCD

- **Expansions in $\alpha_s$:**

  \[
  \sigma_{kl} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sigma_{kl}^{(n)}
  \]

  \[
  H_{ij} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n H_{ij}^{(n)}
  \]

  \[
  \phi_{i/k}(x) = \delta_{ik} \delta(1-x) + \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \phi_{i/k}^{(n)}
  \]

  \[
  \uparrow
  \]

  \[
  \phi_{i/k}^{(0)}(\alpha_s = 0 \Rightarrow \text{Parton } k \text{ " stays itself "}
  \]

- **Consequences:**

  \[
  H_{ij}^{(0)} = \sigma_{ij}^{(0)} \text{ = "Born"}
  \]

  \[
  H_{ij}^{(1)} = \sigma_{ij}^{(1)} - \left[ \sigma_{il}^{(0)} \phi_{i/j}^{(1)} + \phi_{k/i}^{(1)} \sigma_{kj}^{(0)} \right]
  \]

- Suppressed "^" from now on

- **Computed from Feynman diagrams** (process dependent)

- **Computed from the definition of perturbative parton distribution function** (process independent, scheme dependent)
Feynman Rules

- Quark Propagator

\[ \frac{i(p^2 - m^2 + i\epsilon)\delta_{ij}}{p^2 - m^2 + i\epsilon} \quad (i,j=1,2,3) \]

- Gluon Propagator

\[ \frac{i(-g_{\omega})\delta_{ab}}{k^2 + i\epsilon} \quad (a,b=1,2,...,8) \]

- Quark-W Vertex

\[ i \frac{g_w}{\sqrt{2}} (\gamma_\mu)_{\beta\alpha} \left(1 - \gamma_5\right) \frac{1}{2} \delta_{ij} \]

\[ g_w = \frac{e}{\sin\theta_w}, \text{ weak coupling} \]

- Quark-Gluon Vertex

\[ -ig(t_c)_{ji} (\gamma_\mu)_{\beta\alpha} \]

\[ t_c \text{ is the } SU(N)_{N \times N} \text{ generator} \]

- Quark Color Generators

\[ [t_a, t_b] = i f_{abc} t_c \]

\[ \sum_c t^2_c = C_F I_{N \times N} \]

\[ Tr(\sum_c t^2_c) = N C_F \]

\[ C_F = \frac{N^2 - 1}{2N} = \frac{4}{3}, \quad (N = 3) \]
Feynman Diagrams

- Born level \( \alpha_s^{(0)} \) virtual corrections \( (q\bar{q'})_{Born} \)

- NLO: \( \alpha_s^{(1)} \) real emission diagrams \( (q\bar{q'})_{virt} \)

- NLO: \( \alpha_s^{(1)} \) real emission diagrams \( (q\bar{q'})_{real} \)

- NLO: \( \alpha_s^{(1)} \) real emission diagrams \( (\bar{q}G)_{real} \)

- NLO: \( \alpha_s^{(1)} \) real emission diagrams \( (G\bar{q'})_{real} \)
In "Cut-diagram" notation

- $(q\bar{q}')(\text{Born})$

- $(q\bar{q}')(\text{virt})$

- $(q\bar{q}')(\text{real})$

- $(qG)(\text{real})$

- $(G\bar{q}')(\text{real})$

Same as $(qG)_{\text{real}}$ after replacing $q$ by $\bar{q}'$. 
Feynman rules for cut-diagrams

- **Quark line**

\[
\begin{align*}
\nu, \alpha & \quad \frac{k}{\mu, \beta} \\
\nu, \alpha & \quad \frac{p}{j, \beta}
\end{align*}
\]

\[
(2\pi)\delta^+(p^2 - m^2)(\delta + m)_{\beta \alpha} \delta_{ij}
\]

\[
\delta(p^2 - m^2)\theta(p_0)
\]

- **Gluon line**

\[
\begin{align*}
\nu, \alpha & \quad \frac{k}{\mu, \beta} \\
\nu, \alpha & \quad \frac{p}{j, \beta}
\end{align*}
\]

\[
(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}
\]

- **W-boson line**

\[
\begin{align*}
\nu & \quad \frac{q}{\mu} \\
\nu & \quad \frac{p}{j, \beta}
\end{align*}
\]

\[
(2\pi)\delta^+(q^2 - M^2)(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2})
\]

Doesn’t contribute for \( m_f = 0 \) because of Ward identity

- **pp**

\[
\begin{align*}
\nu & \quad \frac{p}{j, \beta} \\
\nu & \quad \frac{p}{j, \beta}
\end{align*}
\]

\[
i \frac{g_W}{\sqrt{2}} \gamma_{\nu \frac{1}{2}} (1 - \gamma_5)
\]

\[
-i \frac{g_W}{\sqrt{2}} \gamma_{\mu \frac{1}{2}} (1 - \gamma_5)
\]
Immediate problems (Singularities)

- **Ultraviolet singularity**

  \[ \int d^4k \frac{k \cdot k}{(k^2)(k^2)(k^2)} \to \infty \]

- **Infrared singularities**

  \[ \frac{1}{(p - k)^2 - m^2} = \frac{1}{-2p \cdot k} \quad (\text{for } m = 0 \text{ or } m \neq 0) \]

  \[ p \cdot k \to 0 \text{ as } \quad \begin{align*}
  k &\to 0 \quad \text{or} \quad k^\mu \parallel p^\mu \quad (\text{for } m = 0) \\
  k &\to 0 \quad (\text{for } m \neq 0)
  \end{align*} \]

  (Similar singularities also exist in virtual diagrams.)

- **Solutions**

  Compute \( H_{ij} \) in pQCD in \( n = 4 - 2\varepsilon \) dimensions

  (dimensional regularization)

  (1) \( n \neq 4 \Rightarrow \text{UV} \& \text{IR} \text{ divergences appear as } \frac{1}{\varepsilon} \text{ poles in } \sigma_{ij}^{(1)} \text{ (Feynman diagram calculation)} \)

  (2) \( H_{ij} \text{ is IR safe } \Rightarrow \text{no } \frac{1}{\varepsilon} \text{ in } H_{ij} \)

      \( (H_{ij} \text{ is UV safe after } "\text{renormalization}".) \)
Dimensional Regularization
(Revisit the Born Cross Section in $n$ dimensions)

\[ \hat{\sigma}_{q\bar{q}'}^{(0)} = \frac{1}{2\tilde{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1}} \frac{\delta^n}{2q_0} (2\pi)^n \cdot \delta^n (p_1 + p_2 - q) \cdot |m|^2 \]

\[ |m|^2 = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot \begin{array}{c}
\text{In } n\text{-dim, the polarization degree of freedom}
\text{ is (2) for a quark, and (n-2) for a gluon.}
\end{array} \]

- Using the Naive-$\gamma^5$ prescription:

\[ Tr \left[ \not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L \right] (-1) \]
\[ = Tr \left[ \not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L \right] (-1) \]
\[ = (-2) (1 - \varepsilon) Tr \left[ \not{p}_1 \not{p}_2 P_L \right] (-1) \]
\[ = (-2) (1 - \varepsilon) \cdot \frac{1}{2} \cdot 4 (p_1 \cdot p_2) (-1) \]
\[ = 2 (1 - \varepsilon) \tilde{s} \]

- In $n$ dimensions

\[ \hat{\sigma}_{q\bar{q}'}^{(0)} = \frac{\pi}{12\tilde{s}} g_w^2 \cdot (1 - \varepsilon) \cdot \delta (1 - \hat{\tau}) \equiv \sigma^{(0)} \cdot \delta (1 - \hat{\tau}) \]
Strong Coupling $g$ in $n$ dimensions

- In $n$ dimensions

$$\int d^n x \mathcal{L} \longrightarrow \int d^n x \left\{ \bar{\psi} i \not \partial \psi - \frac{1}{4} G_{\mu \nu} G^{\mu \nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \cdots \right\}$$

The dimension in mass unit ($\mu$)

$$[\psi] \sim \mu^{\frac{n-1}{2}}$$

$$[G] \sim \mu^{\frac{n-2}{2}}$$

$$[\bar{\psi}G\psi] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

Since $[g\bar{\psi}G\psi] \sim \mu^n$, so

$$[g] \sim \mu^{\frac{-n}{2} + 2} \quad \quad n = 4 - 2\varepsilon$$

$$= \mu^{\varepsilon}$$

$\Rightarrow$ $g$ has a dimension in mass when $\varepsilon \neq 0$

$\Rightarrow$ Feynman rules should read $g \rightarrow g\mu^{\varepsilon}$
Calculations

- Tools needed for a NLO calculation are collected in Appendices A-D

- The detailed calculation for each subprocess can be found in Appendices E

- In the following, I shall summarize the result for each subprocess
Virtual Corrections \((q\bar{q}'_{\text{virt}})\) in Feynman Gauge

- \[ \frac{1}{\varepsilon_{\text{IR}}} \] \[ \frac{1}{\varepsilon_{\text{UV}}} \] and \( \frac{1}{\varepsilon_{\text{IR}}} \) poles cancel when \( \varepsilon_{\text{UV}} = -\varepsilon_{\text{IR}} \equiv \varepsilon \)

- \[ \frac{1}{\varepsilon_{\text{UV}}} \] cancel \( \Rightarrow \) Electroweak coupling is not renormalized by QCD interactions at one-loop order (Ward identity, a renormalizable theory)

- \( \frac{1}{\varepsilon_{\text{IR}}} \) poles remain

\( \sigma^{(1)}_{\text{virt}} \) is free of ultraviolet singularity.

\[
\sigma^{(1)}_{\text{virt}} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta (1 - \frac{1}{\hat{\tau}}) \left( \frac{4\pi\mu^2}{M^2} \right)^\varepsilon \frac{\Gamma (1 - \varepsilon)}{\Gamma (1 - 2\varepsilon)} \cdot \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} \right\} \cdot (C_F)
\]

- \( \frac{2}{\varepsilon^2} \): soft and collinear singularities
- \( \frac{3}{\varepsilon} \): soft or collinear singularities
- \( C_F \): color factor
- \( \sigma^{(0)} \equiv \frac{\pi}{12\delta} g_w^2 \cdot (1 - \varepsilon) \)
Real Emission Contribution \((qq')_{\text{real}}\)

- \(\sim \frac{1}{\varepsilon} \) Collinear

- \(\sim \frac{1}{\varepsilon^2} \) Soft and Collinear

\[
\sigma^{(1)}_{\text{real}} (qq') = \sigma^{(0)} \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{M^2} \right)^\varepsilon \frac{\Gamma (1 - \varepsilon)}{\Gamma (1 - 2\varepsilon)} \cdot C_F \\
\cdot \left\{ \frac{2}{\varepsilon^2} \delta (1 - \tilde{\tau}) - \frac{2}{\varepsilon} \frac{1 + \tilde{\tau}^2}{(1 - \tilde{\tau})^+} + 4 \left( 1 + \tilde{\tau}^2 \right) \left( \frac{\ln (1 - \tilde{\tau})}{1 - \tilde{\tau}} \right)_+ + 2 \frac{1 + \tilde{\tau}^2}{1 - \tilde{\tau}} \ln \tilde{\tau} \right\}
\]

Note: \(\cdots\)_+ is a distribution,

\[
\int_0^1 dz \ f(z) \left[ \frac{1}{1-z} \right]_+ = \int_0^1 dz \frac{f(z) - f(1)}{1-z}, \text{ which is finite.}
\]

- In the soft limit, \(\tilde{\tau} \to 1 \) \((\tilde{\tau} = \frac{M^2}{s})\),

\[
\sigma^{(1)}_{\text{real}} (qq') \to \sigma^{(0)} \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{M^2} \right)^\varepsilon \frac{\Gamma (1 - \varepsilon)}{\Gamma (1 - 2\varepsilon)} \cdot C_F \\
\cdot \left\{ \frac{2}{\varepsilon^2} \delta (1 - \tilde{\tau}) - \frac{4}{\varepsilon} \frac{1}{(1 - \tilde{\tau})^+} + 8 \left( \frac{\ln (1 - \tilde{\tau})}{1 - \tilde{\tau}} \right)_+ \right\}
\]
\[(q\bar{q}')_{\text{virt}} + (q\bar{q}')_{\text{real}} \text{ at NLO}\]

\[\sigma^{(1)}_{q\bar{q}} = \sigma^{(1)}_{\text{virt}}(q\bar{q}) + \sigma^{(1)}_{\text{real}}(q\bar{q}')\]

\[= \sigma^{(0)} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2}\right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \cdot C_F\]

\[\cdot \left\{-\frac{2}{\varepsilon} \left(\frac{1 + \hat{\tau}^2}{1 - \hat{\tau}}\right) - 2\frac{1 + \hat{\tau}^2}{1 - \hat{\tau}} \ln \hat{\tau} + 4\left(1 + \hat{\tau}^2\right) \left(\frac{\ln(1 - \hat{\tau})}{1 - \hat{\tau}}\right) + \left(\frac{2\pi^2}{3} - 8\right) \delta(1 - \hat{\tau})\right\}\]

Where we have used

\[-\frac{2}{\varepsilon} \left[\frac{1 + \hat{\tau}^2}{(1 - \hat{\tau})_+} + \frac{3}{2} \delta(1 - \hat{\tau})\right] = -\frac{2}{\varepsilon} \left(\frac{1 + \hat{\tau}^2}{1 - \hat{\tau}}\right)_+\]

All the soft singularities \(\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)\) cancel in \(\sigma^{(1)}_{q\bar{q}}\)

\[\Rightarrow KLN \text{ theorem}\]

(Kinoshita-Lee-Navenberg)

\[\sigma^{(1)}_{q\bar{q}} \sim \frac{1}{\varepsilon} \text{(term)} + \text{finite (terms)}\]

Collinear Singularity
Factorization Theorem

- Perturbative PDF
  \[ \phi_{i/k}^{(0)} = \delta_{ik}\delta(1-x) \]
  \[ \frac{\alpha_s}{\pi} \phi_{i/k}^{(1)} \] can be calculated from the definition of PDF.

(Process independent, but factorization scheme dependent)

- (1)
  \[ \sigma_{kl}^{(0)} = H_{ij}^{(0)} \Rightarrow H_{kl}^{(0)} = \sigma_{kl}^{(0)} \]

- (2)
  \[ \sigma_{kl}^{(1)} = H_{ij}^{(0)} + H_{ij}^{(0)} + H_{ij}^{(1)} \]

\[ \Rightarrow H_{kl}^{(1)} = \sigma_{kl}^{(1)} - \left[ \phi_{i/k}^{(1)} H_{il}^{(0)} + H_{kj}^{(0)} \phi_{j/l}^{(1)} \right] \]

Finite Divergent
Perturbative PDF

- In $\overline{MS}$-scheme (modified minimal subtraction)

\[
\phi^{(1)}_{q/q}(z) = \phi^{(1)}_{\bar{q}/\bar{q}}(z) = \frac{-1}{2} \left( 4\pi e^{-\gamma_E} \right)^\varepsilon P^{(1)}_{q\rightarrow q}(z)
\]

\[
\phi^{(1)}_{q/g}(z) = \phi^{(1)}_{\bar{q}/\bar{g}}(z) = \frac{-1}{2} \left( 4\pi e^{-\gamma_E} \right)^\varepsilon P^{(1)}_{q\rightarrow g}(z)
\]

where the splitting kernel for $\phi^{(1)}_{q\rightarrow q}$ is

\[
P^{(1)}_{q\rightarrow q}(z) = C_F \left( 1 + z^2 \right) \left( \frac{1 + z^2}{1 - z} + \frac{3}{2} \delta(1 - z) \right),
\]

and for $\phi^{(1)}_{q\rightarrow g}$ is

\[
P^{(1)}_{q\rightarrow g}(z) = T_R \left( z^2 + (1 - z)^2 \right),
\]

where $C_F = \frac{4}{3}$ and $T_R = \frac{1}{2}$.

Note: The Pole part in the $\overline{MS}$ scheme is

\[
\frac{1}{\varepsilon} = \frac{1}{\varepsilon} (4\pi e^{-\gamma_E})^\varepsilon = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E
\]

In the MS scheme, the pole part is just $\frac{1}{\varepsilon}$.
Find $H_{qq'}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$
\sigma_{qq'}^{(1)} = \sigma^{(0)} \left\{ P_{q\bar{q}}^{(1)} (\hat{\tau}) \left[ \ln \left( \frac{M^2}{\mu^2} \right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] + C_F \left[ -\frac{1 + \hat{\tau}^2}{1 - \hat{\tau}} \ln \hat{\tau} + 2 \left( 1 + \tau^2 \right) \left( \frac{\ln (1 - \hat{\tau})}{1 - \hat{\tau}} \right) + \left( \frac{\pi^2}{3} - 4 \right) \delta (1 - \hat{\tau}) \right] \right\}
$$

- $H_{qq'}^{(1)} (\hat{\tau}) = \sigma_{qq'}^{(1)} - \left[ 2 \phi_{q\bar{q}}^{(1)} \sigma_{qq'}^{(0)} \right]$

$$
= \hat{\sigma}^{(0)} \cdot \left\{ P_{q\bar{q}}^{(1)} (\hat{\tau}) \ln \left( \frac{M^2}{\mu^2} \right) + C_F \left[ -\frac{1 + \hat{\tau}^2}{1 - \hat{\tau}} \ln \hat{\tau} + 2 \left( 1 + \tau^2 \right) \left( \frac{\ln (1 - \hat{\tau})}{1 - \hat{\tau}} \right) + \left( \frac{\pi^2}{3} - 4 \right) \delta (1 - \hat{\tau}) \right] \right\}
$$

where

$$
\hat{\tau} = \frac{M^2}{s} = \frac{M^2}{x_1 x_2 S}, \quad \sigma^{(0)} = \hat{\sigma}^{(0)} \cdot (1 - \varepsilon),
$$

$$
\hat{\sigma}^{(0)} = \frac{\pi}{12 s} g_w^2 = \frac{\pi g_w^2}{12 S} \frac{1}{x_1 x_2}.
$$

- pQCD prediction

$$
\sigma_{hh'} = \left\{ \sum_{f=q,\bar{q}} \int dx_1 dx_2 \phi_{f/h} (x_1, \mu^2) \left[ \sigma^{(0)} \delta (1 - \hat{\tau}) \right] \phi_{\bar{f}/h'} (x_2, \mu^2) 
+ \sum_{f=q,\bar{q}} \int dx_1 dx_2 \phi_{f/h} (x_1, \mu^2) \left[ \frac{\alpha_s (\mu^2)}{\pi} H_{f\bar{f}}^{(1)} (\hat{\tau}) \right] \phi_{\bar{f}/h'} (x_2, \mu^2) 
+ \sum_{f=q,\bar{q}} \int dx_1 dx_2 \phi_{f/h} (x_1, \mu^2) \left[ \frac{\alpha_s (\mu^2)}{\pi} H_{fG}^{(1)} (\hat{\tau}) \right] \phi_{G/h'} (x_2, \mu^2) + (x_1 \leftrightarrow x_2) \right\}
$$
Find $H_{qG}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left( \frac{\alpha_s}{\pi} \right)$

$$
\sigma_{qG}^{(1)} = \sigma_{qG}^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2 P_{q\rightarrow g}^{(1)} (\hat{\tau}) \left[ \frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2 (1-\hat{\tau})^2}{4\pi \mu^2 \hat{\tau}} \right] \\
+ \frac{1}{2} + 3\hat{\tau} - \frac{7}{2}\hat{\tau}^2 \right\}
$$

- Similarly,

$$
H_{qG}^{(1)} (\hat{\tau}) = \sigma_{qG}^{(1)} - \left[ \sigma_{qq}^{(0)} \phi_{q'\rightarrow G}^{(1)} \right] \\
= \hat{\sigma}_1^{(0)} \cdot \left\{ \frac{1}{2} \cdot \left[ P_{q\rightarrow g}^{(1)} (\hat{\tau}) \left[ \ln \left( \frac{M^2}{\mu^2} \right) + \ln \left( \frac{(1-\hat{\tau})^2}{\hat{\tau}} \right) \right] \\
+ \frac{1}{4} + \frac{3}{2}\hat{\tau} - \frac{7}{4}\hat{\tau}^2 \right\}
$$

- Similarly,

$$
H_{G\bar{q}}^{(1)} = \sigma_{G\bar{q}}^{(1)} - \left[ \phi_{q'\rightarrow G}^{(1)} \sigma_{qq}^{(0)} \right] \\
= H_{qG}^{(1)}
$$

Note: If we choose the renormalization scale $\mu^2 = M^2$, then $\ln \left( \frac{M^2}{\mu^2} \right) = 0$
W and Z production

* CDF and D0 would like to use their W and Z cross sections for luminosity determination
* D0 cross sections close to center of PDF prediction ellipse; not the case with CDF

• $\phi_{f/h}(x, \mu^2)$ depends on scheme ($\overline{MS}$, DIS,...) 
$\Rightarrow H_{ij}$ scheme dependent

• Evolution equations allow us to predict 
  $q^2$–dependent of $\phi(x, q^2)$

• Essentially identical procedure for 
  $hh' \rightarrow jets$, inclusive $Q\overline{Q},...$
  But, when the Born level process involves strong interaction (eg. $q\overline{q} \rightarrow t\overline{t}$),
it is also necessary to renormalize the strong coupling $\alpha_s$, etc, to eliminate ultraviolate singularities
Appendix A

\(\gamma\)-matrices in \(n\) dimensions

- **Dirac algebra**

\[
\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}
\]

\[
\mu, \nu = 1, 2, \ldots, n \quad g^{\mu\nu} = \text{diag}(1, -1, \ldots, -1)
\]

\[
g^{\mu\nu}g_{\mu\nu} = n
\]

\[
\{\gamma^\mu, \gamma^5\} = 0 \quad (\text{Naive-}\gamma^5\text{prescription})
\]

This works in calculating the inclusive rate of \(W\)-boson, but fails in the differential distributions of the leptons from the \(W\)-boson decay.

- **Matrix identities**

\[
n = 4 - 2\varepsilon
\]

\[
\gamma_\mu \gamma_\mu = -2 (1 - \varepsilon) \gamma_5
\]

\[
\gamma_\mu \gamma_\mu \gamma_\mu = 4a \cdot b - 2\varepsilon \delta_5
\]

\[
\gamma_\mu \gamma_\mu \gamma_\mu \gamma_\mu = -2 \gamma_5 \cdot a + 2\varepsilon \delta_5
\]

- **Traces**

\[
Tr [\gamma_5 \cdot a] = 4(a \cdot b)
\]

\[
Tr [\gamma_5 \cdot a \cdot b] = 4 \{(a \cdot b) (c \cdot d) - (a \cdot c) (b \cdot d) + (a \cdot d) (b \cdot c)\}
\]

\[
Tr [\gamma_5 \cdot a \cdot b] = 0
\]
Appendix B
Some integrals and ”special functions”

• The ”Gamma function”

\[ \Gamma(z) = \int_0^\infty dx x^{z-1}e^{-x} \quad (\text{Re}(z) > 0) \]

\[ \Gamma(z - 1) = \frac{\Gamma(z)}{z - 1} \quad \text{(for all } z) \]

\[ \Rightarrow \text{Poles in } \Gamma(z) \]

\[ \Gamma(n) = (n - 1)! \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

\[ \Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2} \left( \gamma_E^2 + \frac{\pi^2}{6} \right) + \cdots \]

\( (\gamma_E = 0.5772 \cdots, \text{Euler constant}) \)

\[ \Gamma(1 - \varepsilon) = -\varepsilon \Gamma(\varepsilon) = 1 + \varepsilon \gamma_E + \frac{1}{2} \varepsilon^2 \left( \frac{\pi^2}{6} + \gamma_E^2 \right) + O(\varepsilon^3) \]

\[ \Gamma(1 - \varepsilon) \Gamma(1 + \varepsilon) = 1 + \varepsilon^2 \frac{\pi^2}{6} + O(\varepsilon^4) \]

\[ z^\varepsilon = e^{\ln z^\varepsilon} = e^{\varepsilon \ln z} = 1 + \varepsilon \ln z + \cdots \]

• The ”Beta function”

\[ B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \]

\[ B(\alpha, \beta) = \int_0^1 dy y^{\alpha-1} (1 - y)^{\beta-1} = \int_0^\infty dy y^{\alpha-1} (1 + y)^{-\alpha-\beta} \]

\[ = 2 \int_0^{\pi/2} d\theta \left( \sin \theta \right)^{2\alpha-1} \left( \cos \theta \right)^{2\beta-1} \]
• Feynman trick

\[
\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax + b(1-x)]^2}
\]

\[
\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}}
\]

• n-dimension integrals

\[
\int d^n l \frac{l_\mu}{(l^2 - M^2)^\alpha} = 0
\]

\[
\int d^n l \frac{l_\mu l_\nu}{(l^2 - M^2)^\alpha} = \int d^n l \frac{\left(\frac{l^2 g_{\mu\nu}}{n}\right)}{(l^2 - M^2)^\alpha}
\]

\[
\int d^n l \frac{1}{(2\pi)^n (l^2 - M^2)^\alpha} = - \frac{1}{n/2} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)} \left(\frac{1}{M^2}\right)^{\alpha - \frac{n}{2}}
\]

\[
\int d^n l \frac{l^2}{(l^2 - M^2)^\alpha} = \int d^n l \frac{(l^2 - M^2) + M^2}{(l^2 - M^2)^\alpha}
\]

•

\[
\text{Re}[(−1)^\varepsilon] = 1 - \varepsilon^2 \frac{\pi^2}{2} + O(\varepsilon^4)
\]
"plus distribution" — to isolate $\frac{1}{\epsilon}$ poles

Consider

$$\frac{1}{(1-z)^{1+2\epsilon}}$$

$$= \frac{1}{(1-z)^{1+2\epsilon}} - \left[ \delta(1-z) \int_0^1 \frac{dz'}{(1-z')^{1+2\epsilon}} + \frac{1}{2\epsilon} \delta(1-z) \right]$$

because

$$\int_0^1 \frac{dz'}{(1-z')^{1+2\epsilon}} = -\frac{1}{2\epsilon}$$

for $\epsilon \to 0^-$

$$\equiv \left[ \frac{1}{(1-z)^{1+2\epsilon}} \right]_+ - \frac{1}{2\epsilon} \delta(1-z)$$

$$= \frac{1}{(1-z)_+} - 2\epsilon \left[ \frac{\ln(1-z)}{1-z} \right]_+ + O(\epsilon^2) - \frac{1}{2\epsilon} \delta(1-z)$$

because

$$\frac{1}{(1-z)^{2\epsilon}} = (1-z)^{-2\epsilon}$$

$$= 1 - 2\epsilon \ln(1-z) + \cdots$$

$\left[ \cdots \right]_+$ is a distribution

$$\int_0^1 dz \ f(z) \left[ \frac{1}{1-z} \right]_+$$

$$\equiv \int_0^1 dz \ \frac{f(z)}{1-z} - \int_0^1 dz \ f(z) \delta(1-z) \int_0^1 \frac{dz'}{(1-z')}$$

$$= \int_0^1 dz \frac{f(z) - f(1)}{1-z},$$

which is finite.
Appendix C

Angular integrals in \( n \) dimensions

- In \( n \) dimensions

\[
\int d^n x = \int r^{n-1} d\Omega_{n-1}
\]

- \( \int d\Omega = \int_0^\pi d\theta_{n-1} \sin^{n-1} \theta_{n-1} \int_0^\pi d\theta_{n-2} \sin^{n-2} \theta_{n-2} \cdots \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi
\]

\[
\Rightarrow \int d\Omega_1 = \int_0^{2\pi} d\phi \quad \xrightarrow{\text{because } \Omega_1 = 2\pi}
\]

\[
\int d\Omega_2 = \int_0^\pi d\theta_1 \sin \theta_1 \int d\Omega_1 \quad \xrightarrow{\text{because } \Omega_2 = 4\pi}
\]

\[
\vdots
\]

\[
\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1} \theta_{n-1} \int d\Omega_{n-1}
\]

\[
\Rightarrow \Omega_n = \frac{2^n \pi^{\frac{n}{2}} \Gamma \left( \frac{n}{2} \right)}{\Gamma \left( n \right)} \quad \text{from repeated use of } B(\alpha, \beta)
\]

\[
= \frac{2^n \pi^{\frac{n+1}{2}}}{\Gamma \left( \frac{n+1}{2} \right)} \quad \text{because } \Gamma \left( n \right) = \frac{2^{n-1} \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)}
\]
Appendix D

Two-particle phase space in $n$ dimensions

\[
\int_{PS_2(p)} dk \, dq = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1} 2k_0} \frac{d^{n-1}\vec{q}}{(2\pi)^{n-1} 2q_0} \cdot (2\pi)^n \delta^n (p - q - k)
\]

with \( k^\mu = (k_0, \vec{k}) \), etc.

Use \( \frac{d^{n-1}\vec{q}}{2q_0} = \int d^n q \delta^+ (q^2 - Q^2) \), we get

\[
\int_{PS_2(p)} dk \, dq = \frac{1}{(2\pi)^{n-2}} \int \frac{d^{n-1}\vec{k}}{2k_0} \delta^+ ((p - k)^2 - Q^2)
\]

\[
= \frac{1}{(2\pi)^{n-2}} \int \frac{dk \, k^{n-3}}{2} \int d\Omega_{n-2} \delta (\hat{s} - 2k\sqrt{\hat{s}} - Q^2)
\]

\[
(p^2 \equiv \hat{s}, k^2 = 0, k = |\vec{k}|)
\]

Use \( n = 4 - 2\varepsilon \), then in the c.m. frame \( (p^\mu = (\sqrt{\hat{s}}, 0)) \),

\[
\int_{PS_2(p)} dk \, dq = \frac{\Omega_{n-3}}{(2\pi)^2 (1-\varepsilon)} \int \frac{dk \, k^{1-2\varepsilon}}{4\sqrt{\hat{s}}} \int_0^\pi d\theta (\sin \theta)^{1-2\varepsilon} \cdot \delta \left( k - \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}} \right)
\]

Use new variables:

\[
z = \frac{Q^2}{\hat{s}}, y = \frac{1}{2} (1 + \cos \theta) \Rightarrow k = \frac{\sqrt{\hat{s}}}{2} (1 - z),
\]

\[
\int_{PS_2(p)} dk \, dq = \frac{1}{8\pi} \left( \frac{4\pi}{Q^2} \right)^\varepsilon \frac{z^\varepsilon (1 - z)^{1-2\varepsilon}}{\Gamma (1 - \varepsilon)} \int_0^1 dy \left[ y (1 - y) \right]^{-\varepsilon}
\]
Consider

\[
\int \frac{d^nk}{(2\pi)^n} \frac{\gamma_\mu (\not{p} - \not{k}) \gamma_\mu}{(k^2 + i\epsilon) ((p - k)^2 + i\epsilon)}
\]

\[
\rightarrow \int \frac{d^nk}{(2\pi)^n} \int_0^1 dx \frac{(2 - n) (\not{p} - \not{k})}{[k^2 - 2k \cdot xp]^2} \quad (l \equiv k - xp)
\]

\[
= \int \frac{d^nl}{(2\pi)^n} \int_0^1 dx \frac{(2 - n) [(1 - x) \not{p} - \not{l}]}{[l^2 + i\epsilon]^2}
\]

\[
= \left[ \left( 1 - \frac{n}{2} \right) \not{p} \right] \cdot \int \frac{d^nl}{(2\pi)^n} \frac{1}{[l^2 + i\epsilon]^2}
\]

\[
\downarrow \quad = 0 \quad \text{(Because there is no mass scale)}
\]

\[
\uparrow \quad \text{Due to cancellation of } \frac{1}{\varepsilon_{UV}} \text{ and } \frac{1}{\varepsilon_{IR}}
\]

\[
\text{(Trick: } A = A - B + B) \]

\[
= \int \frac{d^nl}{(2\pi)^n} \left\{ \frac{1}{(l^2)^2} - \frac{1}{(l^2 - \Lambda^2)^2} \right\} \quad \text{IR div.} + \frac{1}{(l^2 - \Lambda^2)^2} \quad \text{UV div.}
\]

\[
= \frac{i}{16\pi^2} \left( \frac{1}{\varepsilon_{IR}} \right) + \frac{i}{16\pi} \left( \frac{1}{\varepsilon_{UV}} \right), \quad \left( n - 4 = 2\varepsilon_{IR} \right) \quad \left( 4 - n = 2\varepsilon_{UV} \right)
\]
• consider the real emission process

Define the Mandelstam variables

\[ \hat{s} = (p_1 + p_2)^2 = 2p_1 \cdot p_2 \]
\[ \hat{t} = (p_1 - p_3)^2 = -2p_1 \cdot p_3 \]
\[ \hat{u} = (p_2 - p_3)^2 = -2p_2 \cdot p_3 \]

After averaging over colors and spins

\[
|\mathcal{M}|^2 = \left( \frac{1}{2} \cdot \frac{1}{2} \right) \cdot \left( \frac{1}{3} \cdot \frac{1}{8} \right) \cdot Tr(t^a t^a) \cdot (g\mu^\varepsilon)^2 \\
\cdot g_w^2 \cdot 2(1 - \varepsilon) \\
\cdot \left[ (1 - \varepsilon) \left( -\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}} \right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon \right]
\]

Note: The d.o.f. of gluon polarization is \(2(1 - \varepsilon)\), and that of quark polarization is 2.
In the parton c.m. frame, the constituent cross section
\[ \hat{\sigma} = \frac{1}{2\hat{s}} |M|^2 \cdot (PS_2) \]

\[ = \frac{1}{2\hat{s}} \cdot \left\{ \frac{1}{4} \cdot \frac{1}{6} \cdot 2g_s^2 \mu^{2\varepsilon} g_w^2 (1 - \varepsilon) \cdot \left[ (1 - \varepsilon) \left( -\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}} \right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon \right] \right\} \]

\[ \cdot \left\{ \frac{1}{8\pi} \left( \frac{4\pi}{M^2} \right)^\varepsilon \frac{1}{\Gamma(1 - \varepsilon)} \hat{\tau}^\varepsilon (1 - \hat{\tau})^{1 - 2\varepsilon} \int_0^1 dy \left[ y (1 - y) \right]^{-\varepsilon} \right\} \]

where \( y \equiv \frac{1}{2} (1 + \cos \theta) \)

Using \( \hat{t} = -\hat{s} \left( 1 - \frac{M^2}{\hat{s}} \right) (1 - y) \)

\( \hat{u} = -\hat{s} \left( 1 - \frac{M^2}{\hat{s}} \right) y \)

and

\[ \int_0^1 dy \ y^\alpha (1 - y)^\beta = \frac{\Gamma (1 + \alpha) \Gamma (1 + \beta)}{\Gamma (2 + \alpha + \beta)} , \]

we get

\[ \hat{\sigma}_{qG} = \hat{\sigma}^{(0)} \frac{\alpha_s}{4\pi} \cdot \left\{ 2P_{q\rightarrow g}^{(1)} (\hat{\tau}) \left[ -\frac{1}{\varepsilon} \frac{\Gamma (1 - \varepsilon)}{\Gamma (1 - 2\varepsilon)} + \ln \frac{M^2 (1 - \hat{\tau})^2}{4\pi \mu^2 \hat{\tau}} \right] \right\} , \]

with

\[ P_{q\rightarrow g}^{(1)} (\hat{\tau}) = \frac{1}{2} \left[ \hat{\tau}^2 + (1 - \hat{\tau})^2 \right] \]

\[ \hat{\sigma}^{(0)} = \frac{\pi}{12} g_w^2 \frac{1}{\hat{s}} \]